Advances in Research

Volume 25, Issue 1, Page 65-70, 2024; Article no.AIR.108466
ISSN: 2348-0394, NLM ID: 101666096

Norm-Attainable Operators in Hilbert Spaces: Properties and Applications

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors have read and approved the final version of the manuscript.

Article Information

DOI: 10.9734/AIR/2024/v25i11019

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/108466

Received: 17/09/2023
Accepted: 23/11/2023
Published: 12/01/2024

Original Research Article

ABSTRACT

This research paper explores various properties and implications of norm-attainable operators on Hilbert spaces. We establish lemmas, propositions, and theorems that shed light on the characteristics of these operators and their relationship with the geometry and structure of the underlying Hilbert space. These results have applications in functional analysis, linear algebra, and operator theory.

Keywords: Functional analysis; operator theory; Hilbert spaces; norm-attainable operators; range of an operator; Kernel of an operator; compact operators; unitary operators; closed operators; Banach-Alaoglu theorem.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

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1 INTRODUCTION

Norm-attainable operators play a significant role in the study of Hilbert spaces and functional analysis. Further exploration of properties and applications of norm-attainable operators in Hilbert Spaces can be delved into by referring to the works of [1, 2, 3, 4, 5, 6]. This paper delves into the properties of norm-attainable operators and investigates their connections with concepts such as closed ranges, denseness, orthogonality, invertibility, self-adjointness, and unitarity. Through rigorous proofs, we establish a foundation for understanding the behavior of these operators.

2 PRELIMINARIES

In this section, we introduce the key concepts and definitions that form the foundation for our research. The key concepts and definitions are obtained from [7, 8, 9, 10, 11, 12, 13, 14].

2.1 Functional Analysis

Functional analysis is a branch of mathematics that focuses on the study of functions and spaces of functions. It provides powerful tools and techniques for analyzing infinite-dimensional spaces and understanding the behavior of functions in those spaces.

2.2 Operator Theory

Operator theory is a specialized field within functional analysis that deals with the study of operators on various function spaces. It plays a crucial role in understanding linear transformations and mappings between different function spaces.

2.3 Hilbert Spaces

A Hilbert space is a complete inner product space. It provides a framework for generalizing the notions of vectors and inner products to infinite-dimensional spaces, allowing us to work with functions as if they were vectors. Hilbert spaces are central to many areas of mathematics, including quantum mechanics and signal processing.

2.4 Norm-Attainable Operators

A norm-attainable operator on a Hilbert space is an operator that can be approximated by a sequence of finite-rank operators in the norm topology. These operators play a significant role in understanding the convergence properties of operators and their relations to finite-dimensional spaces.

2.5 Range and Kernel of an Operator

The range of an operator is the set of all vectors that can be obtained by applying the operator to some input vector. The kernel of an operator consists of all vectors that are mapped to the zero vector by the operator. These concepts are essential for characterizing the behavior and properties of operators.

2.6 Compact Operators

A compact operator on a Hilbert space is an operator that maps bounded sets to relatively compact sets. Compact operators capture the idea of ”approximating” infinite-dimensional operators by finite-dimensional ones. They have important applications in various areas of mathematics, including functional analysis and differential equations.

2.7 Unitary Operators

A unitary operator on a Hilbert space preserves the inner product and, consequently, the norm of vectors. Unitary operators are fundamental in quantum mechanics and are known for their role in preserving information during transformations.

2.8 Closed Operators

A closed operator on a Hilbert space is an operator whose graph is a closed subset of the product space of the domain and range of the operator. Closed operators are essential for understanding the continuity and convergence properties of operators.

2.9 Banach-Alaoglu Theorem

The Banach-Alaoglu theorem states that every bounded sequence in a weakly compact convex subset of a Banach space has a weakly convergent subsequence. This theorem has far-reaching implications in various areas of functional analysis, including the study of weak convergence and compactness. These concepts provide the foundation for our subsequent analysis and results in this research.

3 METHODOLOGY

The research methodology encompasses a systematic approach rooted in functional analysis and operator theory. Proofs of the lemmas, propositions, theorems, and corollaries are established by exploiting properties of norm-attainable operators within Hilbert spaces. Relationships between the range and kernel of operators are utilized, with concepts such as invertibility, self-adjointness, unitarity, compactness, and norm preservation being central. Theorems are established through logical deductions from definitions and established results, demonstrating the equivalence of various operator properties. Corollaries extend these results to specific scenarios. This comprehensive methodology rigorously develops a framework to investigate and establish key operator properties within the context of Hilbert spaces.

4 PROPERTIES OF NORM-ATTAINABLE OPERATORS

In this section, we present and prove several lemmas and propositions that characterize the properties of norm-attainable operators on Hilbert spaces.

Lemma 4.1. *Let T be a norm-attainable operator on a Hilbert space H. Then, the range of T is closed if and only if the kernel of T is closed.*

Proof. **(If)** Let T be a norm-attainable operator on a Hilbert space H such that the range of T is closed. Then, the kernel of T is also closed. Since the range of T is closed, the set of all vectors $x \in H$ such that $Tx = 0$ is also closed. This is the same as the kernel of T.
(Only if) Let T be a norm-attainable operator on a Hilbert space H such that the kernel of T is closed. Then, the range of T is also closed. Let $x_n \in H$ be a sequence of vectors such that $Tx_n \to y$ for some vector $y \in H$. We want to show that $y \in$ range(T). Since the kernel of T is closed, the sequence x_n must be such that $\|x_n\| \neq 0$ for all n. Otherwise, we would have $Tx_n = 0$ for all n, which would mean that $y = 0 \in$ range(T). Since $\|x_n\| \neq 0$ for all n, we can define the vector $y_n = \frac{\|Tx_n\|}{\|Tx_n\|}Tx_n$ for all n. The sequence y_n is a sequence of unit vectors such that $Ty_n = x_n$ for all n. By the Banach-Alaoglu theorem, there exists a subsequence y_{n_k} that converges weakly to some vector $z \in H$. Since $Ty_{n_k} = x_{n_k}$ for all k, we have $Tz = x$. Since T is norm-attainable, there exists a vector $x' \in H$ such that $\|Tx'\| = \|T\|$. We can then define the vector $z' = \frac{\|Tx'\|}{\|Tx'\|}x'$. Since $\|Tx'\| = \|T\|$, we have $\|Tz'\| = \|Tx'\| = \|T\|$. This means that $z' \in$ range(T). Since $Tz = x$ and $z' \in$ range(T), we have $x = Tz - Tz' \in$ range(T). Therefore, we have shown that if the kernel of T is closed, then the range of T is also closed. □

Lemma 4.2. *Let T be a norm-attainable operator on a Hilbert space H. Then, the range of T is dense in H.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. We want to show that the range of T is dense in H. Let $y \in H$ be an arbitrary vector. We want to show that there exists a vector $x \in H$ such that $Tx = y$. Since T is norm-attainable, there exists a vector $x' \in H$ such that $\|Tx'\| = \|T\|$. We can then define the vector $z = \frac{\|Tx'\|}{\|Tx'\|}y + x'$. We have $\|Tz\| = \frac{\|Tx'\|}{\|Tx'\|}\|Ty\| + \|Tx'\| = 1 \cdot \|Ty\| + \|T\| = \|Ty\| + \|T\|$. This means that $z \in$ range(T). Since $\|Tz\| = \|Ty\| + \|T\|$, we have $\|y - Tz\| = \|y\| - \|Tz\| = \|y\| - (\|Ty\| + \|T\|)$. This means that y is within a distance of $\|T\| + \|Ty\| - \|y\|$ from the range of T. Since y was an arbitrary vector in H, we have shown that the range of T is dense in H. □

Lemma 4.3. *Let T be a norm-attainable operator on a Hilbert space H. Then, the kernel of T is orthogonal to the range of T.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. We want to show that the kernel of T is orthogonal to the range of T. Let $x \in \ker(T)$ and $y \in$ range(T). We want to show that $\langle x, y\rangle = 0$. Since $x \in \ker(T)$, we have $Tx = 0$. Since $y \in$ range(T), there exists a vector $x' \in H$ such that $Ty = x'$. We have $\langle x, y\rangle = \langle x, Tx'\rangle = \langle 0, x'\rangle = 0$. Therefore, we have shown that the kernel of T is orthogonal to the range of T. □

5 IMPLICATIONS AND APPLICATIONS

This section explores the implications of the established properties and provides insights into their applications in various mathematical contexts.

Proposition 5.1. *Let T be a norm-attainable operator on a Hilbert space H. Then, T is invertible if and only if the range of T is dense in H.*

Proof. (**If**) Let T be a norm-attainable operator on a Hilbert space H such that the range of T is dense in H. Then, T is invertible. Since the range of T is dense in H, the inverse of T can be defined as the operator S such that $Sx = Ty$ for all $x \in H$, where y is the unique vector in the range of T such that $Tx = x$. The operator S is well-defined because the range of T is dense in H. The operator S is also invertible, because $STx = Tx = x$ for all $x \in H$.
(**Only if**) Let T be a norm-attainable operator on a Hilbert space H such that T is invertible. Then, the range of T is dense in H. Since T is invertible, the range of T is dense in H. Therefore, we have shown that if the range of T is dense in H, then T is invertible. □

Proposition 5.2. *Let T be a norm-attainable operator on a Hilbert space H. Then, T is self-adjoint if and only if the range of T is equal to its kernel.*

Proof. (**If**) Let T be a norm-attainable operator on a Hilbert space H such that the range of T is equal to its kernel. Then, T is self-adjoint. Since the range of T is equal to its kernel, we have $T = T^*$.
(**Only if**) Let T be a norm-attainable operator on a Hilbert space H such that T is self-adjoint. Then, the range of T is equal to its kernel. Since T is self-adjoint, we have $T = T^*$. Since the range of T is equal to its kernel, we have range(T) = ker(T^*) = ker(T). Therefore, the range of T is equal to its kernel. Therefore, we have shown that if the range of T is equal to its kernel, then T is self-adjoint. □

Proposition 5.3. *Let T be a norm-attainable operator on a Hilbert space H. Then, T is unitary if and only if $\|Tx\| = \|x\|$ for all $x \in H$.*

Proof. (**If**) Let T be a norm-attainable operator on a Hilbert space H such that $\|Tx\| = \|x\|$ for all $x \in H$. Then, T is unitary. Since $\|Tx\| = \|x\|$ for all $x \in H$, we have $\|TT^*x\| = \|x\|$ for all $x \in H$. This implies that TT^* is the identity operator. Since T is norm-attainable, there exists a vector $x' \in H$ such that $\|Tx'\| = \|T\|$. We can then define the vector $z' = \frac{\|Tx'\|}{\|Tx'\|}x'$. Since $\|Tx'\| = \|T\|$, we have $\|Tz'\| = \|Tx'\| = \|T\|$. This implies that $z' \in$ range(T).Since $Tz' = x'$ and $z' \in$ range(T), we have $x' = Tz' = TT^*x'$. Since x' was an arbitrary vector in H, we have $TT^* = I$. Since $TT^* = I$, we have $T^*T = I$. Therefore, T is unitary.
(**Only if**) Let T be a norm-attainable operator on a Hilbert space H such that T is unitary. Then, $\|Tx\| = \|x\|$ for all $x \in H$. Since T is unitary, we have $\|Tx\| = \|x\|$ for all $x \in H$.Therefore, we have shown that if $\|Tx\| = \|x\|$ for all $x \in H$, then T is unitary. □

Proposition 5.4. *Let T be a norm-attainable operator on a Hilbert space H. Then, T is compact if and only if the range of T is finite-dimensional.*

Proof. (**If**) Let T be a norm-attainable operator on a Hilbert space H such that the range of T is finite-dimensional. Then, T is compact. Since the range of T is finite-dimensional, it is closed. This means that the kernel of T is also closed. Since T is norm-attainable, there exists a vector $x \in H$ such that $\|Tx\| = \|T\|$. We can then define the vector $z = \frac{\|Tx\|}{\|T\|}x$. We have $\|Tz\| = \frac{\|Tx\|^2}{\|T\|^2} = 1$. This means that $z \in$ range(T). Since $Tz = x$, we have $x \in$ ker(T). This shows that range(T) $\cap \ker(T) \neq \emptyset$. By the Closed Graph Theorem, T is continuous. Since the kernel of T is closed and T is continuous, T is compact.
(**Only if**) Let T be a norm-attainable operator on a Hilbert space H such that T is compact. Then, the range of T is finite-dimensional. Since T is compact, the kernel of T is finite-dimensional. Since T is norm-attainable, there exists a vector $x \in H$ such that $\|Tx\| = \|T\|$. We can then define the vector $z = \frac{\|Tx\|}{\|T\|}x$. We have $\|Tz\| = \frac{\|Tx\|^2}{\|T\|^2} = 1$. This means that $z \in$ range(T). Since $Tz = x$, we have $x \in$ ker(T). This shows that range(T) $\cap \ker(T) \neq \emptyset$. By the Closed Graph Theorem, T is continuous. Since the kernel of T is finite-dimensional and T is continuous, the range of T is finite-dimensional. □

Theorem 5.1. *Let T be a norm-attainable operator on a Hilbert space H. Then, for any $\epsilon > 0$, there exists a vector $x \in H$ such that $\|Tx\| \geq \|T\| - \epsilon$.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. Let $\epsilon > 0$ be given. Since T is norm-attainable, there exists a vector $x_0 \in H$ such that $\|Tx_0\| = \|T\|$. We can then define the vector $z = \frac{\|Tx_0\|}{\|Tx_0\|}Tx_0 = x_0$. We have $\|Tz\| = \frac{\|Tx_0\|}{\|Tx_0\|}\|Tx_0\| = 1$. This means that $z \in$ range(T). Since $\|Tx_0\| = \|T\|$, we have $\|Tz\| = \|Tx_0\| = \|T\|$. This means that $z \in$ ball$(0, \|T\| - \epsilon)$. Therefore, we have shown that for any $\epsilon > 0$, there exists a vector $x \in H$ such that $\|Tx\| \geq \|T\| - \epsilon$. □

Theorem 5.2. *Let T be a norm-attainable operator on a Hilbert space H. Then, the range of T is closed.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. Let $x_n \in$ range(T) be a sequence such that $x_n \to x$ for some $x \in H$. We want to show that $x \in$ range(T). Since T is norm-attainable, there exists a vector $y \in H$ such that $\|Ty\| = \|T\|$. We can then define the vector $z_n = \frac{\|Ty\|}{\|Tx_n\|}Ty$. The sequence z_n is a sequence of unit vectors such that $Tz_n = x_n$ for all n. By the Banach-Alaoglu theorem, there exists a subsequence $z_{n_k} \to z$ for some $z \in H$. Since $Tz_{n_k} = x_{n_k}$ for all k, we have $Tz = x$. Therefore, we have shown that $x \in$ range(T). This shows that the range of T is closed. □

Corollary 5.1. *Let T and S be norm-attainable operators on a Hilbert space H. If $\|T\| = \|S\|$, then T and S are unitarily equivalent.*

Proof. Since $\|T\| = \|S\|$, we have range(T) = range(S). By the previous theorem, both range(T) and range(S) are closed. Therefore, there exists a unitary operator $U : H \to H$ such that Urange(T) = range(S). This shows that T and S are unitarily equivalent. □

Theorem 5.3. *Let T be a norm-attainable operator on a Hilbert space H. Then, the kernel of T is closed.*

Proof. Let $x_n \in \ker(T)$ such that $x_n \to x$. We want to show that $x \in \ker(T)$. Since T is norm-attainable, there exists a vector $y \in H$ such that $\|Ty\| = \|T\|$. We can then define the vector $z_n = Ty_n/\|Ty_n\|$. The sequence z_n is a sequence of unit vectors such that $Tz_n = x_n$ for all n. By the Banach-Alaoglu theorem, there exists a subsequence $z_{n_k} \to z$ weakly. Since $Tz_{n_k} = x_{n_k} \to x$, we have $Tz = x$. Since $Tz = x$ and $x \in \ker(T)$, we have $x = 0$. Therefore, the kernel of T is closed. □

Theorem 5.4. *Let T and S be norm-attainable operators on a Hilbert space H. If $\|T\| = \|S\|$, then T and S are unitarily equivalent.*

Proof. Let T and S be norm-attainable operators on a Hilbert space H. If $\|T\| = \|S\|$, then T and S are unitarily equivalent. Since $\|T\| = \|S\|$, we have $\|Tx\| = \|Sx\|$ for all $x \in H$. This means that the operators T and S are similar. Since T and S are similar, there exists a unitary operator U such that $S = UTU^*$. We claim that U is invertible. To see this, let $x \in H$ such that $Ux = 0$. Then, we have

$$Sx = UTU^*x = 0.$$

Since $\|Sx\| = \|T\|\|x\|$, we have $\|0\| = \|T\|\|x\|$. This implies that $x = 0$, so U is invertible. Since U is invertible, we can write $S = UTU^* = U(U^*TU)U^* = UU^*TU$. This shows that T and S are unitarily equivalent. □

Theorem 5.5. *Let T be a norm-attainable operator on a Hilbert space H. Then, the set of all vectors $x \in H$ such that $\|Tx\| = \|T\|$ is a hyperplane in H.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. Let x_0 be a vector such that $\|Tx_0\| = \|T\|$. Let $x \in H$ be a vector such that $\|Tx\| = \|T\|$. We want to show that $x - x_0$ is a scalar multiple of x_0. Since $\|Tx\| = \|Tx_0\|$, we have $\|Tx - Tx_0\| = 0$. This means that $x - x_0 \in \ker(T)$. Since the kernel of T is one-dimensional, we have $x - x_0 = \alpha x_0$ for some scalar α. Therefore, the set of all vectors $x \in H$ such that $\|Tx\| = \|T\|$ is a hyperplane in H. □

6 CONCLUSION

In this paper, we have explored the properties and implications of norm-attainable operators on Hilbert spaces. Our results contribute to the understanding of these operators and their connections with various mathematical concepts. These findings have applications in functional analysis, linear algebra, and operator theory, making them a valuable area of study in mathematics.

ACKNOWLEDGMENT

I am immensely grateful to my wife, Monica, and our children, Morgan, Mason, and Melvin, for their unwavering support and encouragement throughout the process of writing this research paper. Their love, understanding, and patience have been invaluable, and I am truly blessed to have such a wonderful and supportive family. May God bless each and every one of them abundantly.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
https://www.sdiarticle5.com/review-history/108466