Open Access Library Journal
2023, Volume 10, e10986
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Extend Bertrand's Postulate to Sums of Any Primes

Pham Minh Duc

Department of Physics, VNU University of Science, Hanoi, Vietnam
Email: ducphamminh2710@gmail.com

How to cite this paper: Duc, P.M. (2023) Extend Bertrand's Postulate to Sums of Any Primes. *Open Access Library Journal*, **10**: e10986.
https://doi.org/10.4236/oalib.1110986

Received: November 11, 2023
Accepted: December 15, 2023
Published: December 18, 2023

Abstract

According to Bertrand's postulate, we have $p_n + p_n \ge p_{n+1}$. Is it true that for all $n > 1$ then $p_{n-1} + p_n \ge p_{n+1}$? Then $p_n + p_{n-i} > p_{n+j}$ where $n \ge N$, N is a large enough value and i, j are natural numbers?

Subject Areas

Number Theory

Keywords

Bertrand's Postulate, Rosser's Theorem, L'Hospital Rule, Prime Number

1. Introduction

In 1845, Bertrand conjectured what became known as Bertrand's postulate: twice any prime strictly exceeds the next prime [1]. Tchebichef presented his proof of Bertrand's postulate in 1850 and published it in 1852 [2]. It is now sometimes called the Bertrand-Chebyshev theorem. Surprisingly, a stronger statement seems not to be well known, but is elementary to prove: The sum of any two consecutive primes strictly exceeds the next prime, except for the only equality 2 + 3 = 5. After I conjectured and proved this statement independently, a very helpful referee pointed out that Ishikawa published this result in 1934 (with a different proof) [3]. This observation is a special case of a much more general result, Theorem 2, that is also elementary to prove (given the prime number theorem), and perhaps not previously noticed: If p_n denotes the nth prime, $n = 1, 2, 3, \cdots$ with $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$ and if $c_1, c_2, \cdots, c_j$ are natural numbers (not necessarily distinct), and $d_1, d_2, \cdots, d_i$ are positive integers (not necessarily distinct), and then there exists a positive integer N such that $p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_j} > p_{n+d_1} + p_{n+d_2} + \cdots + p_{n+d_i}$ for ll $n \ge N$. We also have another result: If $i < n$ and j are nonnegative integers, then there exists a large enough positive integer N such that, for all $n \ge N$, $p_n + p_{n-i} > p_{n+j}$. We give some numerical results.

2. Main Result

Theorem 1. If $i < n$ and j are nonnegative integers, then there exists a large enough positive integer N such that, for all $n \ge N$, $p_n + p_{n-i} > p_{n+j}$.

Applying Rosser's theorem for all $n \ge 6$, we have

$$n(\ln n + \ln\ln n - 1) < p_n < n(\ln n + \ln\ln n)$$

$$(n+j)\left[\ln(n+j) + \ln\ln(n+j) - 1\right] < p_{n+j} < (n+j)\left[\ln(n+j) + \ln\ln(n+j)\right]$$

For all $n > i + 6$, we have

$$(n-i)\left[\ln(n-i) + \ln\ln(n-i) - 1\right] < p_{n-i} < (n-i)\left[\ln(n-i) + \ln\ln(n-i)\right]$$

Consider the expression

$$A = \frac{n(\ln n + \ln\ln n - 1) + (n-i)\left[\ln(n-i) + \ln\ln(n-i) - 1\right]}{(n+j)\left[\ln(n+j) + \ln\ln(n+j)\right]}$$

We consider the following limit

$$B = \lim_{n \to +\infty} \frac{n(\ln n + \ln\ln n - 1) + (n-i)\left[\ln(n-i) + \ln\ln(n-i) - 1\right]}{(n+j)\left[\ln(n+j) + \ln\ln(n+j)\right]}$$

Taking the ln of the numerator and denominator and applying L'Hospital Rule gives

$$\lim_{n \to +\infty} n(\ln n + \ln\ln n - 1)$$

$$= \lim_{n \to +\infty} \ln n + \ln\ln n - 1 + n\left(\frac{1}{n} + \frac{\frac{1}{n}}{\ln n}\right)$$

$$= \lim_{n \to +\infty} \ln n + \ln\ln n + \frac{1}{\ln n}$$

$$\lim_{n \to +\infty} n\left[\ln(n-i) + \ln\ln(n-i) - 1\right]$$

$$= \lim_{n \to +\infty} \ln(n-i) + \ln\ln(n-i) - 1 + (n-i)\left(\frac{1}{n-i} + \frac{\frac{1}{n-i}}{\ln(n-i)}\right)$$

$$= \lim_{n \to +\infty} \ln(n-i) + \ln\ln(n-i) + \frac{1}{\ln(n-i)}$$

$$\lim_{n \to +\infty} n\left[\ln(n+j) + \ln\ln(n+j)\right]$$

$$= \lim_{n \to +\infty} \ln(n+j) + \ln\ln(n+j) + (n+j)\left(\frac{1}{n+j} + \frac{\frac{1}{n+j}}{\ln(n+j)}\right)$$

$$= \lim_{n \to +\infty} \ln(n+j) + \ln\ln(n+j) + \frac{1}{\ln(n+j)} + 1$$

Then we see

$$B=\lim_{n\to+\infty}\frac{\ln n+\ln\ln n+\frac{1}{\ln n}+\ln(n-i)+\ln\ln(n-i)+\frac{1}{\ln(n-i)}}{\ln(n+j)+\ln\ln(n+j)+1+\frac{1}{\ln(n+j)}}$$

When $n\to+\infty$ then

$$B=\lim_{n\to+\infty}\frac{\ln n+\ln(n-i)}{\ln(n+j)}=\lim_{n\to+\infty}\frac{\ln(n^2-in)}{\ln(n+j)}=+\infty$$

(Because $n^2-in\gg n+j$, for $n\to+\infty$)

Or, for $n\ge N$, N is a large enough positive integer, then $A>1$,

$$\frac{n(\ln n+\ln\ln n-1)+(n-i)\left[\ln(n-i)+\ln\ln(n-i)-1\right]}{(n+j)\left[\ln(n+j)+\ln\ln(n+j)\right]}>1$$

It turns out, $p_n+p_{n-i}\ge p_{n+j}$.

Theorem 2. If $c_1,c_2,\cdots,c_j$ are j nonnegative integers (not necessarily distinct), and $d_1,d_2,\cdots,d_i$ are i positive integers (not necessarily distinct), with $1\le i<j$, then there exists a large enough positive integer N such that, for all $n\ge N$, $p_{n-c_1}+p_{n-c_2}+\cdots+p_{n-c_j}>p_{n+d_1}+p_{n+d_2}+\cdots+p_{n+d_i}$.

Applying Rosser's theorem for all $n\ge 6$, we have

$$(n+d_i)\left[\ln(n+d_i)+\ln\ln(n+d_i)-1\right]<p_{n+d_i}$$
$$<(n+d_i)\left[\ln(n+d_i)+\ln\ln(n+d_i)\right]$$

For all $n>c_j+6$, we have

$$(n-c_j)\left[\ln(n-c_j)+\ln\ln(n-c_j)-1\right]<p_{n-c_j}$$
$$<(n-c_j)\left[\ln(n-c_j)+\ln\ln(n-c_j)\right]$$

Consider the expression

$$C=\frac{\sum_{g=1}^{j}(n-c_g)\left[\ln(n-c_g)+\ln\ln(n-c_g)-1\right]}{\sum_{h=1}^{i}(n+d_h)\left[\ln(n+d_h)+\ln\ln(n+d_h)\right]}$$

We consider the following limit

$$D=\lim_{n\to+\infty}\frac{\sum_{g=1}^{j}(n-c_g)\left[\ln(n-c_g)+\ln(n-c_g)-1\right]}{\sum_{h=1}^{i}(n+d_h)\left[\ln(n+d_h)+\ln(n+d_h)\right]}$$

Taking the ln of the numerator and denominator and applying L'Hospital Rule gives

$$\lim_{n\to+\infty}\sum_{g=1}^{j}(n-c_g)\left[\ln(n-c_g)+\ln\ln(n-c_g)-1\right]$$
$$=\lim_{n\to+\infty}\sum_{g=1}^{j}\ln(n-c_g)+\ln\ln(n-c_g)+\frac{1}{\ln(n-c_g)}$$

$$\lim_{n\to+\infty}\sum_{h=1}^{i}(n+d_h)\left[\ln(n+d_h)+\ln\ln(n+d_h)\right]$$
$$=\lim_{n\to+\infty}\sum_{h=1}^{i}\ln(n+d_h)+\ln\ln(n+d_h)+\frac{1}{\ln(n+d_h)}+1$$

Then we see

$$D = \lim_{n\to+\infty} \frac{\sum_{g=1}^{j} \ln(n-c_g) + \ln\ln(n-c_g) + \frac{1}{\ln(n-c_g)}}{\sum_{h=1}^{i} \ln(n+d_h) + \ln\ln(n+d_h) + \frac{1}{\ln(n+d_h)} + 1}$$

When $n \to +\infty$ then

$$D = \lim_{n\to+\infty} \frac{\sum_{g=1}^{j} \ln(n-c_g)}{\sum_{h=1}^{i} \ln(n+d_h)} = +\infty$$

(Because $\sum_{g=1}^{j} \ln(n-c_g) \gg \sum_{h=1}^{i} \ln(n+d_h)$, for $n \to +\infty$ and $1 \le i < j$)
Or, for $n \ge N$, N is a large enough positive integer, then $C > 1$,

$$\frac{\sum_{g=1}^{j} (n-c_g)\left[\ln(n-c_g) + \ln\ln(n-c_g) - 1\right]}{\sum_{h=1}^{i} (n+d_h)\left[\ln(n+d_h) + \ln\ln(n+d_h)\right]} > 1$$

It turns out, $p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_j} > p_{n+d_1} + p_{n+d_2} + \cdots + p_{n+d_i}$.

3. Concluding Remark

In this short note we have provided the prime number inequality via Rosser and Schoenfeld bounds [4].

Acknowledgements

I thank VNU University of Science for accompanying me.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Bertrand, J. (1845) M'emoire sur le nombre de valeurs que peut prendre une function quand on y permute les lettres qu'elle renferme. *Journal de l'Ecole Royale Polytechnique Cahier*, **30**, 123-140.

[2] Tchebichef, P. (1852) M'emoire sur les nombres premiers. *Journal de Mathématiques pures et appliquées*, **17**, 366-390.

[3] Ishikawa, H. (1934) Über die Verteilung der Primzahlen. *Science Reports of the Tokyo Bunrika Daigaku, Section A*, **2**, 27-40.

[4] Rosser, J.B. and Schoenfeld, L. (1962) Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, **6**, 64-94. https://doi.org/10.1215/ijm/1255631807