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Two-Dimensional Manifolds with Computation *V*-Function of ODE Systems

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Abstract

The computation of stable or unstable manifold of two-dimensional is developed, which is an efficient method in studying stable structure analysis of system character geometrically. The Lorentz stable manifold is computed by the fixed arclength method and the hyperbolic equilibrium is a saddle. The two-dimensional stable structure of Lorentz manifold is significant in people's usual view. We also introduce the *V*-function to compute the *V*-manifold correspondingly. The defined *V*-function is smooth in the unstable direction of the manifold. Especially, the routh to period-doubling attractor on manifold surface is discussed too.

Keywords

Stable Manifold, Unstable Manifold, V-Function, Attraction Boundary

1. Introduction

The manifold is significant in people's view to give some structural features of system in geometry. Usually, the two-dimensional stable manifold of an unstable hyperbolic equilibrium is formed and the attraction basin is constructed appropriately. The attraction boundary is diversity in dealing with different geometrical characters, for example, a limit cycle, doubly period limit cycle, even multiple period cycle or chaos [1] [2] [3] [4]. The manifold appearing with homoclinic orbit boundary is still few in papers. However, sometimes the heteroclinic orbits growing from two different equilibrium solution brings up with twins manifold [2] [3].

The system dimension has the close relationship with manifold structure. We always consider the two-dimensional manifold of Lorentz system while system has three state variables and an equilibrium solution is hyperbolic saddle. The manifold built algorithm is also applied to a neuron system further, which has an saddle equilibrium solution. Furthermore, we also built the defined *V*-manifold with the appropriate *V*-function. The corresponding *V*-function with definition originated from Lyapunov *V*-function to derive the stability criterion of equilibrium solution. General *V*-function is a simple polynomial and the whole drevivation of *V*-function with respect to time t is computed regularly to analyze the stability of equilibrium solution. And by this way, we in fact expand the original system into four-dimensional space of state variable (x, y, z, V), which insight us to draw two-dimensional manifold surface in four-dimensional state variable system in the future work.

The manifold algorithms have been developed novelty with its aims to overcome the difficult in computing the different increasing rate of arc length with different direction. That is to say, the manifold is usually grown up inhomogenously in different direction. Therefore, people have developed all possible methods as his/her potential handtools. All sorts of the manifold algorithm are rigorous with ambitious to complete its computer realization. We apply the computation method here known as Krauskopf etc. [5] [6] [7], and develop the manifold computation as boundary value problems (BVPs) with the test condition

$$g(x, y, z, T) = 0 \tag{1}$$

In general, we choose fixed arclength with supposed condition

$$\int_T \Gamma(x, y, z, T) = \text{constant} \tag{2}$$

wherein Γ is a fragment solution orbits on manifold surface, or fixed time condition

$$T = \text{constant} \tag{3}$$

We define *V*-function and compute the *V*-manifold in accordance with axis transformation since the built manifold in *x*-*y*-*z* view. *V*-function is defined smoothly to get only simple terms in Lie-derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$ which is to compute the differential of V with respect to time t.

Definition 1: Suppose *S*-manifold is the stable manifold of the saddle equilibrium solution. If there exists a C^∞ function $V: R^3 \to R$, which is positive, and the Lie derivation of V with respect to t is always negative along *S*-manifold, then the function $V(x, y, z)$ is called as the stable *V*-function manifold.

Definition 2: Suppose *U*-manifold is the unstable manifold of the unstable saddle focus. If there exists a C^∞ function $V: R^3 \to R$, and the Lie derivation of V with respect to t is positive along *U*-manifold, then the function $V(x, y, z)$ is called as the unstable *V*-function manifold.

The *V*-function built scheme also gives people the insights to compute two-dimensional manifold in ODEs with high-dimensional and weak nonlinear features. The route to manifold boundary is discussed with designed strategy, alike the control condition $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.

The whole paper is organized as follows. In section 2, as often appearing, the manifold of "butterfly chaos" as lorentz system is computed. In section 3, two-dimensional manifold of a neuron network model is computed and the V-manifold is built too. In section 4, the route to manifold boundary is discussed with the aids of V-derivative with the supposed V-function. The discussion is given finally.

2. The Computation of Stable Manifold of Lorentz System

Lorentz attractor has attracted peoples more attention by 'butterfly chaos' in screen with chosen parameters. In general, Lorentz attractor chaos are often discussed and have infinite unstable periodic oscillating solutions imbedding are still a challenge work. The different initial values with small perturbation among them can bring solution orbits with definitely distance divergence. With given initial values, the chaos is observed, which looks like a butterfly. Always as a typical model example, people built two-dimensional stable manifold of Lorentz system, see papers [5] for reference. The lorentz manifold computation scheme contrast with each other the computer realization efficiency, therefore, the manifold algorithm is developed [7].

The system of Lorentz attractor is written as follows,

$$\begin{cases} \dot{x}_1 = a\left(x_2 - x_1\right), \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = -\beta x_3 + x_1 x_2. \end{cases} \tag{4}$$

with parameters $a = 10, \beta = 8/3, \rho = 28$. Given any initial condition near the trivial equilibrium solution, the "butterfly chaos" is seen by the well-posed problem of ODE system (4). It is easily computed that the trivial equilibrium solution has three eigenvalues, respectively $\lambda = -2.67, -22.8, 11.8$. Therefore, the two-dimensional stable manifold can be defined locally as

$$\begin{aligned} W^s\{0\} &= \left\{x \in R^3 \mid \lim_{t\to+\infty} \phi(t,x) = 0\right\} \\ W^u\{0\} &= \left\{x \in R^3 \mid \lim_{t\to-\infty} \phi(t,x) = 0\right\} \end{aligned} \tag{5}$$

The corresponding linearized system of Equation (4) is written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ \rho - x_3 & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6}$$

With the assumption that the stable eigenspace E^s is spanned by two eigenvectors e_1 and e_2, that is, $E^s = span\{e_1, e_2\}$, the initial design of circle C_0 lying on the stable manifold which encircle zero is represented by

$$C_0 = r_0 \cos\left(2\pi \frac{N}{M}\right) e_1 + r_0 \sin\left(2\pi \frac{N}{M}\right) e_2 \tag{7}$$

since based on the fundamental knowledge, the stable manifold is tangent to the stable eigenspace at zero. With radius r_0, formula (7) design the initial circle

C_0. The next step is to compute a series of circle C_n to expand the stable manifold by adaption of foliation algorithm, which is introduced by Krauskopf in his paper [5] [6] and applied in the computation algorithm in papers [7]. The curvature is calculated with the limitation of lower and upper bound, $\alpha_{\min}$ and $\alpha_{\max}$ respectively, and the distance among circles is assumed to be the constant Δ. At the n-th step, with the chosen parameter τ, the initial point $x_0 = C_n(i-1) + \tau(C_n(i) - C_n(i-1))$ lying inside between nodes $C_n(i-1)$ and $C_n(i)$ of the mesh $C_n = \{C_n(1), \cdots, C_n(i-1), C_n(i), \cdots, C_n(M)\}$, we further compute a fragment of solution curves by ODE-solver, alike ODE23, ODE45 etc., which composed of the stable manifold. Therefore, we should solve the corresponding BVP problem to get the next cricle C_{n+1}. As shown in **Figure 1**, the initial nine circles is calculated which form the impression of local stable manifold. With the total arclength assumed to equal 120, the stable manifold from different view is shown in **Figures 2(a)-(d)**.

3. Manifestation of *V*-Function

Consider the following neuron network model

$$\begin{aligned} x' &= -0.1x + 0.1(1.49\tanh(x) + 2\tanh(y) + \tanh(z)) \\ y' &= -0.1y - 0.2\tanh(x) + 0.17\tanh(y) \\ z' &= -0.1z + 0.4\tanh(x) - 0.4\tanh(y) + 0.2\tanh(z) \end{aligned} \tag{8}$$

and an optional choice for the V-function which is defined as

$$V(x, y, z) = x^2/2 + y^2/2 + z^2/2 + xz + yz + xy \tag{9}$$

The trivial solution of Equation (5) is an unstable saddle. The eigenvalues are 0.2836, −0.0323 ± 0.1806. The corresponding eigen vectors $\{\Re\{e_0\}, \Im\{e_0\}\}$ of

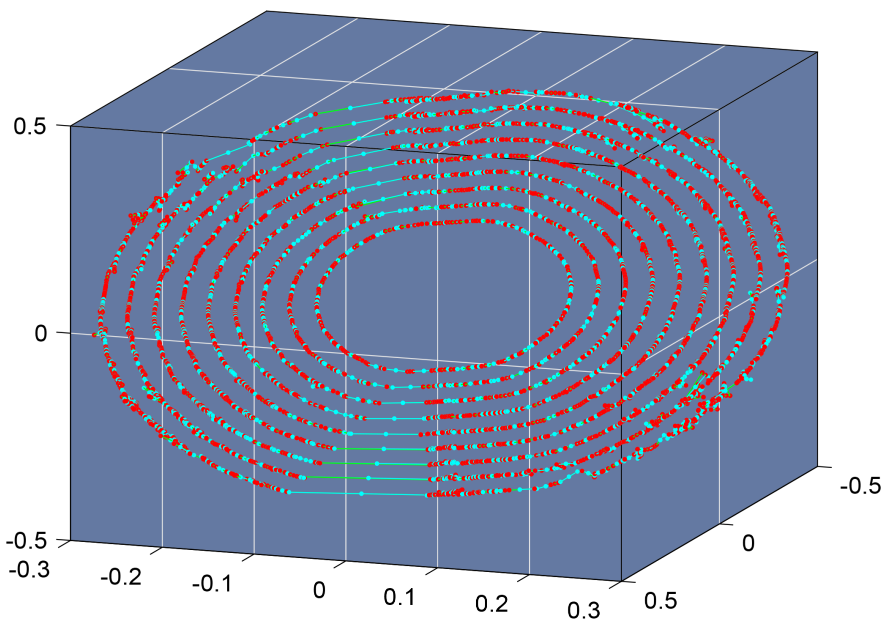

Figure 1. The beginning of manifold computation of Lorentz model.

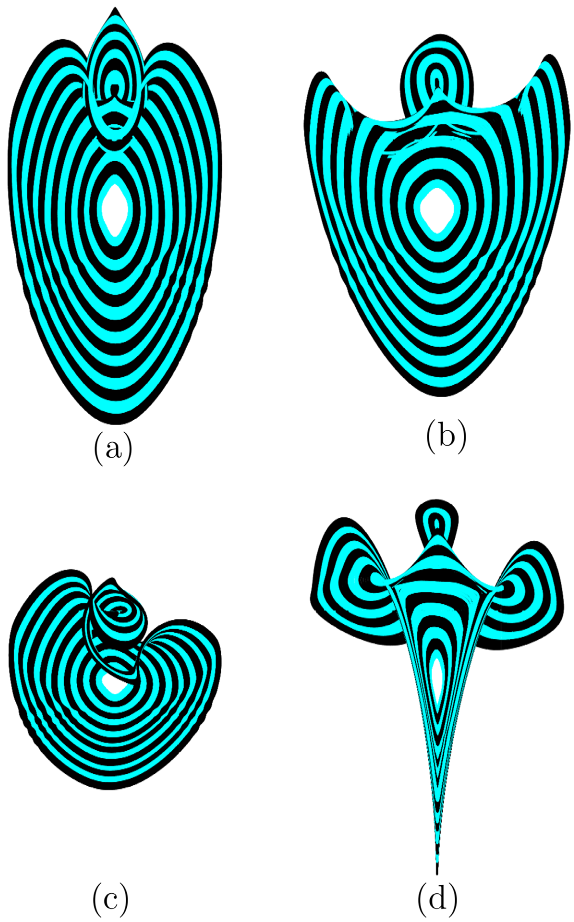

Figure 2. The different sight view of manifold of Lorentz model.

negative eigenvalues compose of the stable eigen subspace. The tangency of the stable manifold to the stable eigen subspace is significant which give our sights that

$$C_0 = r_0 \cos\left(2\pi \frac{N}{M}\right) \Re\{e_0\} + r_0 \sin\left(2\pi \frac{N}{M}\right) \Im\{e_0\} \tag{10}$$

for $N = 0,1,2,\cdots,M-1$. With fixed time step, the next circles are calculated one by one which adapt for the foliation algorithm which is seen [1] [7] for reference. We get the series of circles $C_i\left(i = 0,1,2,\cdots,n\right)$ which expand the two-dimensional stable manifold. However, a challenge work is often choosing time step by $\text{time} = 0.3/\|f\|$ which is most favorable for the circles laterly. The corresponding V-manifold is also plotted in X-Y-V sight. As shown in **Figure 3**, the different view of stable manifold in X-Y-Z sight are observed in **Figure 3(a)** & **Figure 3(b)** and V-manifold are manifested in X-Y-V sight in **Figure 3(c)** & **Figure 3(d)**.

4. Special Route to Periodical Solutions

How to get to the boundary attraction basin if coming from the unstable equilibrium? One efficient way is through the unstable manifold arriving attraction

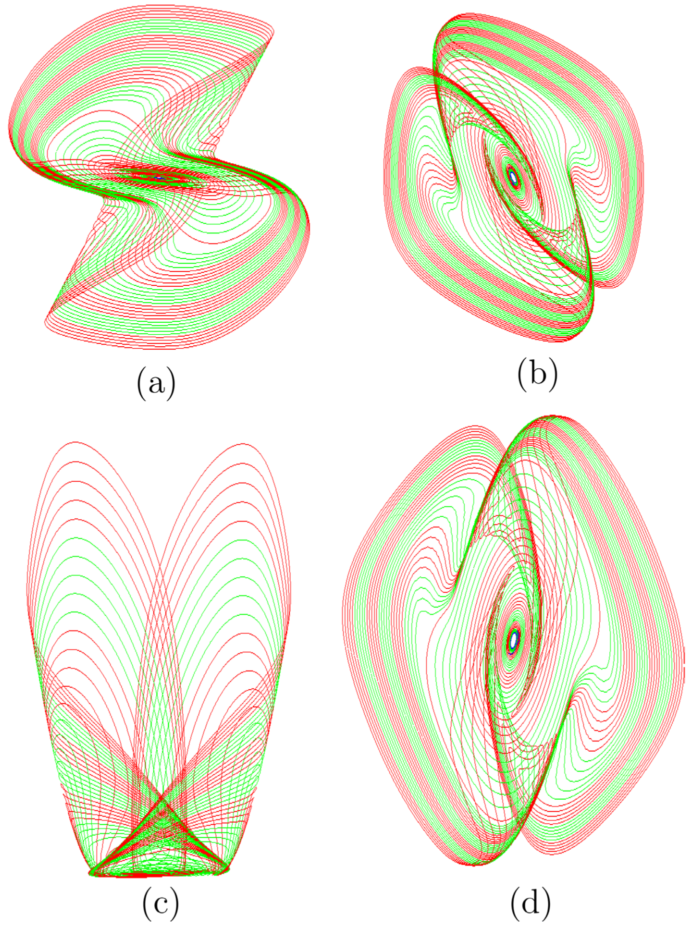

Figure 3. The different sight view of manifold of neuron model.

boundary which forms the neighborhood of the observed doubly period cycle. It sounds interesting to choose an efficient route which spends the shortest time to arrive the boundary. We try to be trustable to choose a V-function with test condition alike $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ etc. The thoughts are originated from the simple way to get the key controlling point in every circle on manifold. Hence after, we choose the following oscillator as an example,

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\alpha z - \beta y - x + x^2 \end{aligned} \tag{11}$$

with $\alpha = 0.6448, \beta = 0.98$. It is computed that the trivial solution is unstable. We compute the two-dimensional stable manifold with limit cycle as boundary. The bifurcating doubly period limit cycle as varying free parameter α is observed, as shown in **Figure 4(a)**, **Figure 4(b)** and **Figure 4(d)**. As shown in **Figure 4(a)**, the stable manifold which has tangency with the stable eigen subspace is expanded and grow into Mobius brand which encircle the doubly period limit cycle again. The direction on Mobius brand is changed continuously, and how to grow manifold is manifested by a special route through manifold surface.

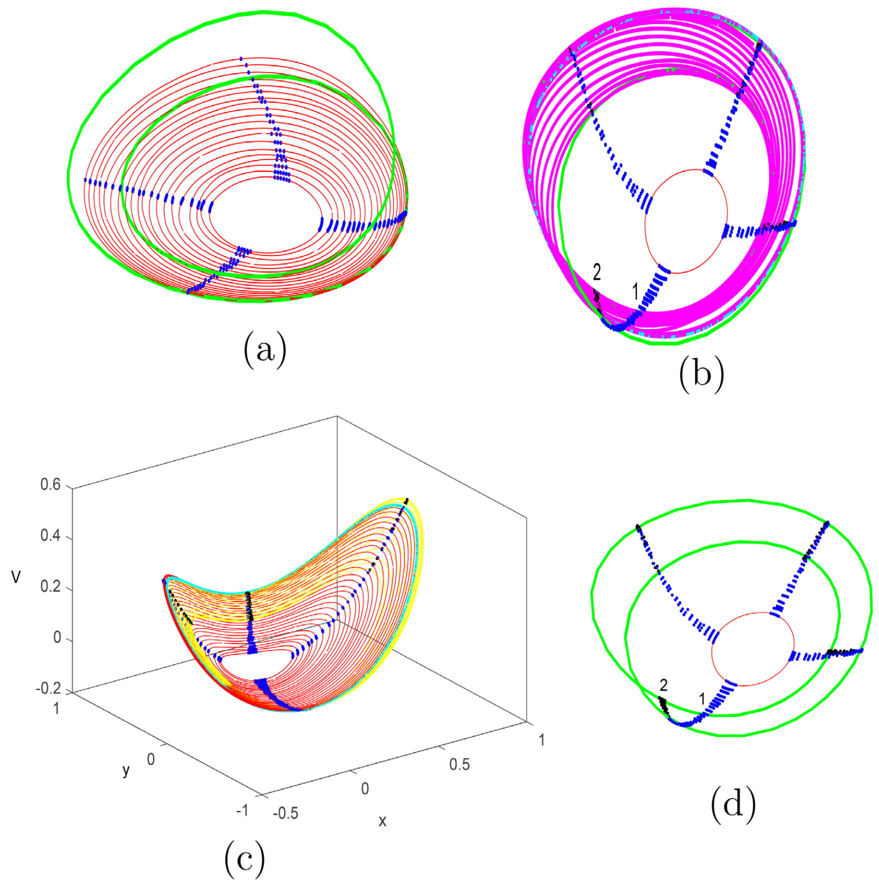

Figure 4. The special route to doubly period solution of manifold of model (8).

The route is denoted by blue points then black points again. As shown in **Figure 4(b)**, it is expanded to the outer boundary of Mobius brand by blue route then folded back again by black route to arrive its inner boundary. The clear image of walking route and doubly period limit cycle is shown in **Figure 4(d)**. The V-manifold is drawn in **Figure 4(c)** and the corresponding V-function is written as

$$V = x^2/2 - x^3/3 + \alpha/2\, y^2 + bxy + yz \tag{12}$$

and the route is computed by setting $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.

5. Discussion

The fundamental knowledge of local manifold is set up by the tendency of solution orbits at infinity and is often divided into stable manifold and unstable manifold whenever the equilibrium solution is hyperbolic. We listed three examples to compute the two-dimensional stable manifold by foliation manifold algorithm with adaption time step. The V-function was defined and V-manifold was discussed which insight us to further compute the stable manifold of 4-dimensional or high-dimensional ODEs. The manifold computation gives people a direct view of growing manifold surface and extremely promotes enthusiasm to shape the attraction basin of hyperbolic equilibrium solution by known algorithm.

Ethics Approval and Consent to Participate

The author declares there's no confliction of ethics approval and consent to participate.

Consent for Publication

Not applicable.

Availability of Data and Materials

All data generated or analysed during this study are included in this published article.

Authors' Contributions

The independent authors solely finish the whole numerical simulation work and the manuscript writing.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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