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Integration of the *n*-th order Linear Differential Equations with Coefficients with Variable Exponential Solutions

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Author's contribution

Author KAK designed the study, and wrote the first draft of the manuscript. Ideas, methods and proofs of solutions of LDE carried out by KAK solely. Author read and approved the final manuscript.

Research Article

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ABSTRACT

This paper covers not linear differential equations (LDE) with variable coefficients but respective Riccati type equations which play a similar role to a characteristic equation during integration of LDE with constant coefficients. We have established a certain analogy of problems of integration of LDE in quadratures with a problem of solution to algebraic equations with radicals [5,6,7,8]. Necessary and sufficient condition for existence of an $e^{\lambda x}$ form solution to an LDE of the n-th order with variable coefficients has been found. At the end of this paper we give specific examples. The solutions of this method can be used in the studies of properties of thermal conductivity, hydrophobicity of composite materials, development of new technologies multilayer asphalt and three-layer wall panel of heterogeneous materials.

Keywords: Characteristic equations of Riccati type; exponential solutions.

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1. INTRODUCTION

The $(n-1)$-th order Riccati type characteristic equation

$$[p+r(x)]^{n-1}r(x)+b_{n-1}(x)[p+r(x)]^{n-2}r(x)+\cdots+ \\ +b_1(x)r(x)+b_0(x)=0, \qquad p=\frac{d}{dx}. \tag{1}$$

Here $[p+r(x)]^k \cdot r(x)$ means the consistent application of the operator k times $[p+r(x)]$ to the function $r(x)$.

Contains an unknown function $r(x)$ in the n-th degree, then it has exactly n 'roots', i.e. solutions, which, of course, may contain constants as particular cases. That is to say LDE with variable coefficients has a solution of $e^{\lambda x}$ form [8]. It is necessary to develop methods of finding constant roots of a characteristic equation of Riccati type.

The invariant subspace method is refined to present more unity and more diversity of exact solutions to evolution equations. The key idea is to take subspaces of solutions to linear ordinary differential equations as invariant subspaces that evolution equations admit. A two-component nonlinear system of dissipative equations is analyzed to shed light on the resulting theory, and two concrete examples are given to find invariant subspaces associated with 2nd-order and 3rd-order linear ordinary differential equations and their corresponding exact solutions with generalized separated variables [1,2].

Each step asks for a particular solution of a Riccati differential equation. These Riccati equations appear to be the generalization of the classical characteristic equation for linear time-invariant systems. As linear time-varying (LTV) systems are concerned, characteristic equations can be obtained using the as is well known, the variational equations of nonlinear dynamic systems are linear time-varying (LTV) by nature. In the modal solutions for these LTV equations, the earlier introduced dynamic eigenvalues play a key role. They are closely related to the Lyapunov- and Floquet-exponents of the corresponding nonlinear systems. In this contribution, we present some simple examples for which analytic solutions exist. It is also demonstrated by example how the classical linear time-invariant (LTI) solutions are related to the equilibrium points of the general LTV solutions [3,4].

2. RESEARCH

Indeed, linear differential equations with variable coefficients

$$y^{(n)}+b_{n-1}(x)y^{(n-1)}+...+b_1(x)y'+b_0(x)y=0,\ b_{i-1}(x)\in C(a,b), \tag{2}$$

are common and they have one (or several) solution(s) of $e^{\lambda x}$ form. In this case, one particular solution to the $(n-1)$-th order Riccati type characteristic equation is a constant $r_1(x)=\lambda=const.$

Constant λ can be complex. We have proved the following theorem related to the allocated equations

Theorem 1: For the linear differential equation (2) to have a solution of $e^{\lambda x}$ form, it is necessary and sufficient if number λ satisfies characteristic equation (1), meanwhile equation (2) reduces to:

$$\begin{gathered}(y'-\lambda y)^{(n-1)}+[\lambda+b_{n-1}(x)](y'-\lambda y)^{(n-2)}+\\+[\lambda^2+b_{n-1}(x)\lambda+b_{n-2}(x)](y'-\lambda y)^{(n-3)}+...+[\lambda^{n-2}+b_{n-1}(x)\lambda^{n-3}+\\+...+b_3(x)\lambda+b_2(x)](y'-\lambda y)'+\\+[\lambda^{n-1}+b_{n-1}(x)\lambda^{n-2}+...+b_2(x)\lambda+b_1(x)](y'-\lambda y)=0.\end{gathered} \tag{3}$$

Proof Necessity: Let's assume that a particular solution to equation (2) is $y=e^{\lambda x}$, i.e. substituting $e^{\lambda x}$ in (2) and dividing it by an exponent yields

$$\lambda^n+b_{n-1}(x)\lambda^{n-1}+...+b_1(x)\lambda+b_0(x)=0, \tag{4}$$

this proves the theorem.

Sufficiency: Suppose that condition (4) is satisfied and let's multiply it term wise by $e^{\lambda x}$. As we know a formula for the *k*-th derivative of an exponent we see that the function satisfies the equation $y=e^{\lambda x}$ (2).

And now let's find new coefficients of an equation reduced by one order, i.e. let's transform differential equation (2) to a form which must be proved (3). For this purpose instead of $b_0(x)$ in equation (2) we substitute its value from (4)

$$\begin{gathered}y^{(n)}+b_{n-1}(x)y^{(n-1)}+...+b_2(x)y''+b_1(x)y'-\\-\lambda y\sum_{i=1}^{n}b_i(x)\lambda^{i-1}=0,\ b_n(x)=1\ \forall x\in(a,b).\end{gathered}$$

Adding and subtracting expression

$$y'\cdot\sum_{i=1}^{n}b_i(x)\lambda^{i-1},$$

in the above equation yield

$$\begin{gathered}y^{(n)}+b_{n-1}(x)y^{(n-1)}+...+b_2(x)y''+b_1(x)y'-y'\sum_{i=2}^{n}b_i(x)\lambda^{i-1}-\\-b_1(x)y'+y'\sum_{i=1}^{n}b_i(x)\lambda^{i-1}-\lambda y\sum_{i=1}^{n}b_i(x)\lambda^{i-1}=0.\end{gathered}$$

Annihilating members $b_1(x)y'$ we export λ as a sign of the first sum, then group the second and the third sums and transform the equation to the following:

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_3(x)y''' + b_2(x)y'' - \lambda y' \sum_{i=2}^{n} b_i(x)\lambda^{i-2} +$$

$$+ (y' - \lambda y)\sum_{i=1}^{n} b_i(x)\lambda^{i-1} = 0.$$

Adding and subtracting expression

$$y'' \sum_{i=2}^{n} b_i(x)\lambda^{i-2},$$

in the above equation yield

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_3(x)y''' + b_2(x)y'' - y'' \sum_{i=3}^{n} b_i(x)\lambda^{i-2} -$$

$$- b_2(x)y'' + y'' \sum_{i=2}^{n} b_i(x)\lambda^{i-2} - \lambda y' \sum_{i=2}^{n} b_i(x)\lambda^{i-2} + (y' - \lambda y)\sum_{i=1}^{n} b_i(x)\lambda^{i-1} = 0.$$

Annihilating terms $b_2(x)y''$ we export λ as a sign of the first sum, then group the second and the third sums, we bring to a derivative sign and transform the equation to the following

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_4(x)y^{(4)} + b_3(x)y''' - \lambda y'' \sum_{i=3}^{n} b_i(x)\lambda^{i-3} -$$

$$+ (y' - \lambda y)' \sum_{i=2}^{n} b_i(x)\lambda^{i-2} + (y' - \lambda y)\sum_{i=1}^{n} b_i(x)\lambda^{i-1} = 0$$

This operation is performed $(n-1)$ times in the above-mentioned manner and yields equation (3).

Consequence 1: For the equation (2) to have a solution of $e^{\lambda x}$ form, it is necessary to have such nonzero number as $\alpha_1, \alpha_2, ..., \alpha_{n-1}$, so that the following condition is met

$$b_0(x) + \alpha_1 b_1(x) + ... + \alpha_{n-1} b_{n-1}(x) = const \neq 0. \tag{5}$$

The proof is obvious. This condition is remarkable because by means of coefficients of a differential equation it can be easily determined if it has a particular solution of $e^{\lambda x}$ form.

Consequence 2: If an algebraic equation of the *n*-th order

$$r^n + b_{n-1}r^{n-1} + b_{n-2}r^{n-2} + ... + b_1 r + b_0 = 0$$

has a root λ, it is represented as follows

$$\begin{aligned}(r-\lambda)[r^{n-1} + (\lambda + b_{n-1})r^{n-2} + (\lambda^2 + b_{n-1}\lambda + b_{n-2})r^{n-3} + ... +\\ + (\lambda^{n-2} + b_{n-1}\lambda^{n-3} + b_{n-2}\lambda^{n-4} + ... + b_3\lambda + b_2)r +\\ + (\lambda^{n-1} + b_{n-1}\lambda^{n-2} + b_{n-2}\lambda^{n-3} + ... + b_2\lambda + b_1)] = 0.\end{aligned} \tag{6}$$

It is being proved similarly to the theorem by adding and subtracting expressions, where the degree r^k is taken instead of derivative $y^{(k)}(x)$.

Theorem 2: If characteristic equation

$$r^n(x) + b_{n-1}(x)r^{n-1}(x) + ... + b_1(x)r(x) + b_0(x) = 0 \tag{7}$$

has ($n-1$) constant roots $\lambda_1, \lambda_2, ..., \lambda_{n-1}$, and functional root λ(*t*). Then linear differential equation (2) becomes a non-homogeneous equation with constant coefficients

$$y^{(n-1)} + a_{n-2}y^{(n-2)} + ... + a_1 y' + a_0 y = C_n \exp \int_{x_0}^{x} \lambda(t)dt, \tag{8}$$

where

$$a_0 = (-1)^{n-1}\lambda_1\lambda_2 \cdot ... \cdot \lambda_{n-1},$$

$$a_1 = (-1)^{n-2}(\lambda_1\lambda_2 \cdot ... \cdot \lambda_{n-2} + \lambda_1\lambda_2 \cdot ... \cdot \lambda_{n-3}\lambda_{n-1} + ... + \lambda_2\lambda_3 \cdot ... \cdot \lambda_{n-1}),$$

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$$a_{n-2} = -(\lambda_1 + \lambda_2 + ... + \lambda_{n-1}).$$

It is quite difficult to solve functional algebraic equation (7) and especially to transform and to extract constant roots which existence is assumed. It is therefore necessary to simplify the process of finding roots. We assume existence of solution to LDE with variable coefficients in $e^{\lambda x}$ form. Equation (7) is true for ∀ $x \in (a,b)$.

In order to find its constant roots it is necessary to consider algebraic equation yielded from (1), where $x = x_1 \in (a,b)$, i.e.

$$r^n + b_{n-1}(x_1)r^{n-1} + \ldots + b_1(x_1)r + b_0(x_1) = 0,$$

which in turn has n roots and the equation itself (7) has less constant roots because of a variability of coefficients of differential equation (2). If we want to determine a solution to algebraic functional equation (7) from a set of constant roots it is necessary to take n points of the interval (a, b). In fact, let's write down equation (7) at various points for solution λ_1:

$$\begin{cases} \lambda_1^n + b_{n-1}(x_1)\lambda_1^{n-1} + \ldots + b_1(x_1)\lambda_1 = -b_0(x_1), \\ \lambda_1^n + b_{n-1}(x_2)\lambda_1^{n-1} + \ldots + b_1(x_2)\lambda_1 = -b_0(x_2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1^n + b_{n-1}(x_n)\lambda_1^{n-1} + \ldots + b_1(x_n)\lambda_1 = -b_0(x_n). \end{cases}$$

The resultant system of n equations with respect to n numbers λ_1^n, λ_1^{n-1},…, λ_1, has an unambiguous solution if a system determinant is different from zero.

First method: If we want to determine particular solutions to LDE (2), it is necessary to find constant roots of equation (7) after it is written down in point x_1 (eg, $b_0(x_1) = 0$ or $b_0(x_1) = b_1(x_1) = \ldots = b_k(x_1) = 0$, $k = 0, 1, 2, \ldots, n-1$) suitable for calculation. From constant roots it is necessary to choose the λ_j which satisfies the functional equation (7), then solutions will be functions $y_j = e^{\lambda_j x}$.

Second method: First let's write down functional algebraic equation (7) at n points $x_\nu \in (a,b)$, $\nu = 1,2,\ldots,n$ and find roots of these equations. If a set of these numbers includes numbers simultaneously being roots of all equations, they can be solutions to functional equation (7), i.e. functions $y_j(x) = e^{\lambda_j x}$ are particular solutions to LDE with variable coefficients (2).

Cases when constants r_k = const exist among variable roots $r_j(x)$ of an algebraic functional equation are indirectly covered by studies [8].

1. $y'' + \frac{a}{x^2} \cdot y' - \left(b^2 + \frac{ab}{x^2}\right) y = 0,\ \ x \neq 0,\ \ y_1(x) = e^{bx}$, $\lambda_1 = b$.
2. $y'' - xy' + (x-1)y = 0,\ \ y_1(x) = e^x$, $\lambda_1 = 1$.
3. 3. $4xy'' + 4y' - (x+2)y = 0,\ \ y_1(x) = e^{\frac{x}{2}}$, $\lambda_1 = 0{,}5$
4. $y'' + y' \cdot \mathrm{tg}\, x - (\alpha \mathrm{tg} x + \alpha^2) y = 0,\ \ \ y_1(x) = e^{\alpha x}$.
5. $y''' - \frac{2x}{x^2+2} y'' - y' + \frac{2x}{x^2+2} y = 0,\ \ y_1 = e^x,\ \ y_2 = e^{-x}$, $\lambda=1$, $\lambda=-1$.
6. $(x^2 - 2x + 3)y''' - (x^2+1)y'' + 2xy' - 2y = 0,$

   $\lambda=1$, $\lambda=-2$, $y_1 = e^x,\ \ y_2 = e^{-2x}$.
7. $y''' - \frac{1}{x}y'' + y' - \frac{1}{x}y = 0,\ \ r(x) = \pm i,\ \ y_1 = \cos x,\ \ \ y_2 = \sin x\cdot$
8. $y''' - y'' \cdot \mathrm{ctg} 3x + 4y' - 4y \cdot \mathrm{ctg} 3x = 0,$

   $r(x) = \pm 2i,\ \ y_1 = \cos 2x,\ \ y_2 = \sin 2x$.

9. $y''' - (x^2 - 4)y'' + (5 - 4x^2)y' + 5x^2y = 0,$

$$r(x) = 2 \pm i, \quad y_1 = e^{2x} \cdot \cos x, \quad y_2 = e^{2x} \cdot \sin x.$$

3. CONCLUSION

In this way we have developed and substantiated the algebraic method for integration of one class of LDE of the *n* – th order with variable coefficients in the presence of constant roots of the (*n*-1)-th order Riccati characteristic equation. We have demonstrated a method of finding constant roots of the above-mentioned equation which can be transformed to an algebraic functional equation coefficients of which can be taken at any point or at different point's .Examples have been given for illustration.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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