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Estimation of Pareto Parameters Using a Fuzzy Least-Squares Method and Other Known Techniques with a Comparison

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Abstract

The purpose of this paper is to obtain the fuzzy least-squares estimator for the two-parameter Pareto distribution and to compare the fuzzy estimator with different types of estimators. The trimmed linear moments (TL-moments), linear moments (L-moments) and linear quantile moments (LQ-moments) formulas will be obtained for the two-parameter Pareto distribution and the TL-moments estimator, L-moments estimator and LQ-moments estimator will be derived for the Pareto distribution. Numerical comparisons between the proposed method and the existing methods are implemented. According to these comparisons, it is suggested that the proposed fuzzy least-squares estimator is preferable all times.

Keywords: Pareto distribution, fuzzy least-squares, TL-moments, L-moments, LQ-moments, maximum likelihood, simulations.

1 Introduction

The Pareto family of life distributions has been found to provide good models in many empirical studies. The Pareto distribution was first proposed as a model for the distribution of incomes. It is also used as a model for the distribution of city populations within a given area. The cumulative distribution function of the Pareto distribution is defined by the following:

$$F(x;\alpha,k)=\begin{cases}1-\left(\dfrac{k}{x}\right)^{\alpha}, & k\le x<\infty\\ 0\ , & -\infty<x<k.\end{cases} \tag{1-1}$$

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where $k > 0$ and $\alpha > 0$ are referred to as the scale and shape parameters. The probability density function is:

$$f(x;\alpha,k) = \begin{cases} \dfrac{\alpha k^{\alpha}}{x^{\alpha+1}}, & k \le x < \infty \\ 0\ , & -\infty < x < k. \end{cases} \tag{1-2}$$

The corresponding quantile function of the Pareto distribution as follows:

$$Q(u) = k(1-u)^{-\frac{1}{\alpha}}, \qquad 0 < u < 1,\ k,\alpha > 0 \tag{1-3}$$

and the corresponding r^{th} moment about zero is:

$$\mu_r = \begin{cases} \infty, & \alpha \le r \\ \dfrac{\alpha k^{r}}{(\alpha - r)}, & \alpha > r,\ k > 0. \end{cases} \tag{1-4}$$

The expected value is:

$$E(X) = \begin{cases} \infty, & \alpha \le 1 \\ \dfrac{\alpha k}{(\alpha - 1)}, & \alpha > 1,\ k > 0. \end{cases} \tag{1-5}$$

and the variance is:

$$Var(X) = \begin{cases} \infty, & \alpha \in (1,\ 2] \\ \dfrac{\alpha k^{2}}{(\alpha - 1)^2(\alpha - 2)}, & \alpha > 2,\ k > 0. \end{cases} \tag{1-6}$$

Hung and Liu [1] introduced a fuzzy least-squares method to estimate the parameters of the Weibull distribution when outliers are present as a robust estimation method. Numerical comparisons between this fuzzy least-squares algorithm and existing methods (the least-squares, the weighted least-squares, the least absolute deviation and Drapella and Kosznik [2]) are implemented. According to these comparisons, they suggested that the proposed fuzzy least-squares algorithm is preferable when the sample size is large.

Hosking [3] introduced the concept of the linear moments (L-moments) and concluded that L-moments of a probability distribution to be meaningful, we require only that the distribution has a finite mean; for standard errors of L-moments to be finite, we require only that the distribution has a finite variance; and L-moments, being linear functions of the data, are less sensitive than are classical moments to sampling variability or measurement errors in the extreme data values. Elamir and Seheult [4] introduced the trimmed linear moments (TL-moments) and concluded that TL-moments are more resistant to outliers, TL-Moments assign zero weight to the extreme observations, they are easy to compute and a population TL-Moments may be well defined where the corresponding population L-Moments (or central moment) does not exist.

Mudholkar and Hutson [5] introduced the concept of the linear quantile moments (LQ-moments) and concluded that LQ-moments are often easier to evaluate and estimate than L-moments, LQ-moments always exist and unique and their asymptotic distributions are easier to obtain. Abu El-Magd [6] obtained the TL-moments and LQ-moments estimators of the exponentiated generalized extreme value distribution. She introduced a numerical simulation compares TL-moments estimators with other estimation methods (L-moments estimators, LQ-moment estimators and the method of moment estimators) mainly with respect to their biases and root mean squared errors.

The main aim of this paper is to introduce the fuzzy least-squares method to estimate the parameters of the Pareto distribution and introduce the TL-moments and LQ-moments of the Pareto distribution. This is a relevant problem because of the usefulness of the Pareto distribution in different applications especially in life testing and reliability theory. The fuzzy least-squares estimators (FLSEs), the TL-moment estimators (TLMEs), L-moments estimators (LMEs), LQ-moment estimators (LQMEs) and the maximum likelihood estimators (MLEs) for the Pareto distribution will be obtained. A numerical simulation compares these methods of estimation mainly with respect to their biases and root mean squared errors (RMSEs) will be obtained.

The remaining sections are as follows. In section two, the maximum likelihood estimator (MLEs), the TL-moments and the LQ-moments with different special cases for the Pareto distribution will be derived. Also, the TL-moments estimators (TLMEs), L-moment estimators (LMEs) and the LQ-moments estimators (LQMEs) will be obtained for the Pareto distribution. In section three, the fuzzy least-squares estimators (FLSEs) will be obtained for the Pareto distribution. In section four, a numerical simulation to compare the properties of the MLEs, TLMEs, LMEs, LQMEs and the FLSEs of the Pareto distribution will be obtained. Finally, the results and conclusion of the numerical comparison between different estimators for the Pareto distribution will be introduced.

2 Estimation of Parameters

We are interested in estimating the parameters of the Pareto distribution from which a random sample comes. This paper will also consider some other relatively new techniques for estimating the required parameters.

2.1 Maximum Likelihood Estimators

The likelihood function, L, for a sample $(x_1, x_2, ..., x_n)$ of the Pareto distribution has the following form:

$$l(k,\alpha)=\prod_{i=1}^{n}\frac{\alpha k^{\alpha}}{x_i^{\alpha+1}}=\frac{\alpha^n k^{\alpha n}}{\left(\prod_{i=1}^{n}x_i\right)^{\alpha+1}}, \qquad k\le x<\infty,\ k,\alpha>0 \tag{2-1}$$

and taking logarithms,

$$L=\log l(k,\alpha)=n\log\alpha+n\alpha\log k-(\alpha+1)\sum_{i=1}^{n}\log x_i \tag{2-2}$$

Hence

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}+n\log k-\sum_{i=1}^{n}\log x_i \tag{2-3}$$

And equating to zero, then the maximum estimator of α will be as follows:

$$\alpha^*=\frac{n}{\sum_{i=1}^{n}\log\frac{x_i}{k^*}} \tag{2-4}$$

A maximum likelihood estimate cannot be obtained for k by differentiating L with respect to k since L is unbounded with respect to k. But since k is lower bound of the random variable x, maximize L subject to the constraint $k^*\le\min_i x_i$. Clearly l is maximized with respect to k subject to the constraint $k^*\le\min_i x_i$ when:

$$k^*=\min_i x_i \tag{2-5}$$

which is, therefore, the maximum likelihood estimate for k.

2.2 TL-Moments and L-Moments Estimators

In this section, the trimmed linear moments (TL-moments) of the two-parameter Pareto distribution will be obtained. From the TL-moments with generalized trimmed, many special cases can be obtained such as the TL-moments with the first trimmed and linear moments (L-moments) for the Pareto distribution. Also, the use of the TL-moments and L-moments for estimating the unknown parameters of the Pareto distribution will be derived.

TL-Moments:

Let, $X_1, X_2, ..., X_n$ be a conceptual random sample (used to define a population quantity) of size n from a continuous distribution and let, $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ denote the corresponding order statistics. Elamir and Seheult [4] defined the r^th TL-moment $\lambda_r^{(s,t)}$ as follows:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} E\left(X_{(r+s-j:r+s+t)}\right), \quad r=1,\ 2,\ 3,\ .., \qquad s,\ t=0,\ 1,\ 2,\ 3,\ .., \tag{2-6}$$

where s, t = 0, 1, 2,…. The TL-moments reduce to L-moments (see Hosking [3]) when s = t = 0. They considered the symmetric case ($s=t$). Hosking [7]obtained some theoretical results for the TL-moments with generalized trimmed for s and t (symmetric case ($s=t$) and asymmetric case ($s \ne t$)) and obtained the TL-moments coefficient of variation TL-CV, the TL-skewness and the TL-kurtosis as follows:

$$\tau^{(s,t)} = \lambda_2^{(s,t)} \big/ \lambda_1^{(s,t)},\ \ \tau_3^{(s,t)} = \lambda_3^{(s,t)} \big/ \lambda_2^{(s,t)},\ \ and\ \ \tau_4^{(s,t)} = \lambda_4^{(s,t)} \big/ \lambda_2^{(s,t)}. \tag{2-7}$$

Maillet and Médecin [8] introduced the relation between the r^th TL-moments and the first TL-moments with generalized trimmed for s and t (symmetric case $(s=t)$ and asymmetric case $(s \ne t)$). Indeed, it is sufficient to compute TL-moments of order one to obtain all TL-moments. They obtained the following r^th TL-moments:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \lambda_1^{(r+s-j-1,t+j)}, \quad r=1,\ 2,\ 3,\ .., \qquad s,\ t=0,\ 1,\ 2,\ 3,\ .., \tag{2-8}$$

where s, t = 0, 1, 2,…. This relation is very important and helped to enable easier calculations for the r^th TL-moments with any trimmed and L-moments as particular cases of the r^th TL-moments with generalized trimmed for s and t. Here, we obtain the r^th TL-moments for the Pareto distribution for $\alpha(t+j+1) \ge 1$ as follows:

$$\lambda_r^{(s,t)} = \frac{k}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \frac{(r+s+t)!}{(r+s-j-1)!(t+j)!} \mathrm{B}\left(r+s-j, t+j+1-1/\alpha\right), \tag{2-9}$$

$$r=1,\ 2,\ 3,\ .., \quad s,\ t=0,\ 1,\ 2,\ 3,\ .., \quad ,\ k,\alpha>0, \quad \alpha\left(t+j+1\right) \ge 1$$

According to the above relation (2-9), the first four TL-moments with generalized trimmed for s and t (s, t = 0, 1, 2, …,) of the Pareto distribution will be:

$$\lambda_1^{(s,t)} = k\frac{(s+t+1)!}{(s)!(t)!}\mathrm{B}\left(s+1,\ t+1-\frac{1}{\alpha}\right), \alpha(t+1)\geq 1 \tag{2-10}$$

$$\lambda_2^{(s,t)} = \frac{k}{2}\left[\frac{(s+t+2)!}{(s+1)!(t)!}\mathrm{B}\left(s+2,\ t+1-\frac{1}{\alpha}\right) - \frac{(s+t+2)!}{(s)!(t+1)!}\mathrm{B}\left(s+1,\ t+2-\frac{1}{\alpha}\right)\right], \alpha(t+2)\geq 1 \tag{2-11}$$

$$\lambda_3^{(s,t)} = \frac{k}{3}\left[\frac{(s+t+3)!}{(s+2)!(t)!}\mathrm{B}\left(s+3,\ t+1-\frac{1}{\alpha}\right) - 2\frac{(s+t+3)!}{(s+1)!(t+1)!}\mathrm{B}\left(s+2,\ t+2-\frac{1}{\alpha}\right) + \frac{(s+t+3)!}{(s)!(t+2)!}\mathrm{B}\left(s+1,\ t+3-\frac{1}{\alpha}\right)\right], \alpha(t+3)\geq 1 \tag{2-12}$$

and

$$\lambda_4^{(s,t)} = \frac{k}{4}\left[\frac{(s+t+4)!}{(s+3)!(t)!}\mathrm{B}\left(s+4,\ t+1-\frac{1}{\alpha}\right) - 3\frac{(s+t+4)!}{(s+2)!(t+1)!}\mathrm{B}\left(s+3,\ t+2-\frac{1}{\alpha}\right) + 3\frac{(s+t+4)!}{(s+1)!(t+2)!}\mathrm{B}\left(s+2,\ t+3-\frac{1}{\alpha}\right) - \frac{(s+t+4)!}{(s)!(t+3)!}\mathrm{B}\left(s+1,\ t+4-\frac{1}{\alpha}\right)\right],\ \alpha(t+4)\geq 1 \tag{2-13}$$

From these results we can obtain the TL- coefficient of variation $\tau^{(s,t)}$, TL-skewness $\tau_3^{(s,t)}$ and TL-kurtosis $\tau_4^{(s,t)}$ for the Pareto distribution.

Special Cases:

a) *The TL-Moments with the first trimmed (* $s = t = 1$*):*

By substituting $s = 1$, and $t = 1$ in equations (2-10), (2-11), (2-12) and (2-13), the first four TL-moments with the first trimmed of the Pareto distribution will be:

$$\lambda_1^{(1)} = 6k\mathrm{B}\left(2, 2-\frac{1}{\alpha}\right), \quad 2\alpha \geq 1 \tag{2-14}$$

$$\lambda_2^{(1)} = 6k\left[\mathrm{B}\left(3, 2-\frac{1}{\alpha}\right) - \mathrm{B}\left(2, 3-\frac{1}{\alpha}\right)\right], \quad 3\alpha \geq 1 \tag{2-15}$$

$$\lambda_3^{(1)} = \frac{20k}{3}\left[B\left(4,2-\frac{1}{\alpha}\right) - 3B\left(3,3-\frac{1}{\alpha}\right) + B\left(2,4-\frac{1}{\alpha}\right)\right], \quad 4\alpha \geq 1 \tag{2-16}$$

And

$$\lambda_4^{(1)} = \frac{15k}{2}\left[B\left(5,2-\frac{1}{\alpha}\right) - 6B\left(4,3-\frac{1}{\alpha}\right) + 6B\left(3,4-\frac{1}{\alpha}\right) - B\left(2,5-\frac{1}{\alpha}\right)\right], \quad 5\alpha \geq 1 \tag{2-17}$$

From these results we can obtain the TL- coefficient of variation $\tau^{(1,1)}$, TL-skewness $\tau_3^{(1,1)}$ and TL-kurtosis $\tau_4^{(1,1)}$ with the first trimmed for the Pareto distribution.

b) The L-Moments ($s = t = 0$):

By substituting $s = 0$, and $t = 0$ in the rth TL-moments for the Pareto distribution, we can obtain the rth L-moments for the Pareto distribution as follows:

$$\lambda_r = \frac{k}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \frac{r!}{(r-j-1)!(j)!} B(r-j, j+1-1/\alpha), \quad r = 1,\ 2,\ \ldots, \quad \alpha(j+1) \geq 1 \tag{2-18}$$

Also, we can obtain the first four L-moments for the Pareto distribution by substituting $s = 0$, and $t = 0$ in equations (2-10), (2-11), (2-12) and (2-13), as a special case from the TL-moments for the Pareto distribution. The first four L-moments for the Pareto distribution will be:

$$\lambda_1 = kB\left(1,1-\frac{1}{\alpha}\right), \quad \alpha \geq 1 \tag{2-19}$$

$$\lambda_2 = k\left[B\left(2,1-\frac{1}{\alpha}\right) - B\left(1,2-\frac{1}{\alpha}\right)\right], \quad 2\alpha \geq 1 \tag{2-20}$$

$$\lambda_3 = k\left[B\left(3,1-\frac{1}{\alpha}\right) - 4B\left(2,2-\frac{1}{\alpha}\right) + B\left(1,3-\frac{1}{\alpha}\right)\right], \quad 3\alpha \geq 1 \tag{2-21}$$

and

$$\lambda_4 = k\left[B\left(4,1-\frac{1}{\alpha}\right) - 9B\left(3,2-\frac{1}{\alpha}\right) + 9B\left(2,3-\frac{1}{\alpha}\right) - B\left(1,4-\frac{1}{\alpha}\right)\right], \quad 4\alpha \geq 1 \tag{2-22}$$

and from the first four L-moments, we can obtain the L-coefficient of variation $\tau = \lambda_2/\lambda_1$, L-skewness $\tau_3 = \lambda_3/\lambda_2$ and L-kurtosis $\tau_4 = \lambda_4/\lambda_2$ for the Pareto distribution.

TL-Moments Estimators:

The TL-moment estimators (TLMEs) for the unknown parameters of the Pareto distribution can be obtained by equating the first two population TL-moments ($\lambda_1^{(s,t)}, \lambda_2^{(s,t)}$) to the corresponding sample TL-moments ($l_1^{(s,t)}$, $l_2^{(s,t)}$) for the Pareto distribution. Hosking [7] obtained the first two sample TL-moments to be:

$$l_1^{(s,t)} = \frac{1}{\binom{n}{s+t+1}} \sum_{j=s+1}^{n-t} \binom{j-1}{s}\binom{n-j}{t} x_{(j:n)}, \qquad (2\text{-}23)$$

and

$$l_2^{(s,t)} = \frac{1}{2\binom{n}{s+t+2}} \sum_{j=s+1}^{n-t} \binom{j-1}{s}\binom{n-j}{t}\left(\frac{(j-s-1)}{(s+1)} - \frac{(n-j-t)}{(t+1)}\right) x_{(j:n)} \qquad (2\text{-}24)$$

Clearly, sample TL-moments reduce to sample L-moments when s = t = 0. Now, we can obtain the TL-moment estimators (TLMEs) ($\hat{\alpha}$ and $\hat{k}$) of the Pareto distribution by solving the following two equations:

$$l_1^{(s,t)} = \hat{k}\frac{(s+t+1)!}{(s)!(t)!} \mathrm{B}\left(s+1,\ t+1-\frac{1}{\hat{\alpha}}\right), \qquad (2\text{-}25)$$

and

$$l_2^{(s,t)} = \frac{\hat{k}}{2}\left[\frac{(s+t+2)!}{(s+1)!(t)!} \mathrm{B}\left(s+2,\ t+1-\frac{1}{\hat{\alpha}}\right) - \frac{(s+t+2)!}{(s)!(t+1)!} \mathrm{B}\left(s+1,\ t+2-\frac{1}{\hat{\alpha}}\right)\right], \qquad (2\text{-}26)$$

The equations (2-25) and (2-26) are valid for any trimmed s and t and. To solve these equations, determine the value of trimmed or the value of s and t; but the resulting equations are difficult to solve (because the beta function is a function of α). So, these equations will be solved numerically. As a special case, by putting s = t = 1, the TLMEs $\hat{\alpha}$ and $\hat{k}$ for the TL-moments with the first trimmedand for s = t = 0, the L-moments estimates (LMEs) can be obtained for the Pareto distribution.

L-Moments Estimators:

Now, we will introduce the L-moment estimators (LMEs) for the Pareto distribution. If $X_{(1:n)} \le X_{(2:n)} \le \ldots \le X_{(n:n)}$ denotes the order sample, we have the first and second sample L-moments as:

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i:n)}, \tag{2-27}$$

And

$$l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i:n)} - l_1. \tag{2-28}$$

Equating the first two population L-moments λ_1, λ_2 to the corresponding sample L-moments l_1, l_2, we will obtain:

$$l_1 = k^{**}\mathrm{B}\left(1, 1-\frac{1}{\alpha^{**}}\right), \tag{2-29}$$

and

$$l_2 = k^{**}\left[\mathrm{B}\left(2, 1-\frac{1}{\alpha^{**}}\right) - \mathrm{B}\left(1, 2-\frac{1}{\alpha^{**}}\right)\right]. \tag{2-30}$$

Then, the LMEs of α and k, say α^{**} and k^{**}, respectively, can be obtained by solving the equations for (2-29) and (2-30).

2.3 LQ-Moments Estimators

In this section, the linear quantile moments (LQ-moments) of the two-parameter Pareto distribution will be obtained with three different cases (median, trimean and Gastwirth). Also, the use of the LQ-moments with three different cases for estimating the unknown parameters of the Pareto distribution will be derived.

LQ-Moments:

Let, $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution function $F(x)$ with quantile function $Q_X(u) = F_X^{-1}(u)$ and let, $X_{(1:n)} \le X_{(2:n)} \le \ldots \le X_{(n:n)}$ denote the order statistics. Mudholkar and Hutson [5] defined the r^{th} population linear quantile moments (LQ-moments) ζ_r of X, as:

$$\zeta_r = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \tau_{p,d}(X_{(r-j:r)}), \qquad r = 1,\ 2,\ 3,\ \ldots \tag{2-31}$$

where $0 \le d \le 1/2,\ 0 \le p \le 1/2$, and

$$\tau_{p,d}(X_{(r-j:r)}) = pQ_{X_{(r-j:r)}}(d) + (1-2p)Q_{X_{(r-j:r)}}(1/2) + pQ_{X_{(r-j:r)}}(1-d). \tag{2-32}$$

The linear combination $\tau_{p,d}$ is a 'quick' measure of the location of the sampling distribution of the order statistic $X_{(r-j:r)}$. The candidates for $\tau_{p,d}$ include the function generating the common quick estimators by using the median ($p = 0.5,\ d = 0.5$), the trimean ($p = 1/4,\ d = 1/4$) and the Gastwirth ($p = 0.3,\ d = 1/3$). They introduced the LQ-skewness and LQ-kurtosis for the population by $\eta_3 = \zeta_3/\zeta_2$ and $\eta_4 = \zeta_4/\zeta_2$ respectively; it may be used for identifying the population and estimating the parameters. The LQ-skewness takes the value of zero for symmetrical distributions.

The LQ-moments with the three cases (median, trimean and Gastwirth) will be obtained for the Pareto distribution as follows:

(1) Using the median ($p = 0.5,\ d = 0.5$), and the quantile function for the Pareto distribution, the first four LQ-moments for the Pareto distribution will be:

$$\xi_1 = k[Q_\circ(0.5)], \tag{2-33}$$

$$\xi_2 = \frac{k}{2}[Q_\circ(0.707) - Q_\circ(0.293)], \tag{2-34}$$

$$\xi_3 = \frac{k}{3}[Q_\circ(0.794) - 2Q_\circ(0.5) + Q_\circ(0.206)], \tag{2-35}$$

and

$$\xi_4 = \frac{k}{4}[Q_\circ(0.841) - 3Q_\circ(0.614) + 3Q_\circ(0.386) - Q_\circ(0.159)]. \tag{2-36}$$

where

$$Q_\circ(u) = (1-u)^{-\frac{1}{\alpha}}, \qquad 0 < u < 1,\ \ \alpha > 0. \tag{2-37}$$

(2) Using the trimean ($p = 1/4,\ d = 1/4$), the first four LQ-moments for the Pareto distribution will be obtained as follows:

$$\xi_1 = \frac{k}{4}\left[Q_\circ(0.25) + 2Q_\circ(0.5) + Q_\circ(0.75)\right], \tag{2-38}$$

$$\xi_2 = \frac{k}{8}\left[Q_\circ(0.866) + 2Q_\circ(0.707) - 2Q_\circ(0.293) - Q_\circ(0.134)\right], \tag{2-39}$$

$$\begin{aligned}\xi_3 = \frac{k}{12}\big[&Q_\circ(0.909) + 2Q_\circ(0.794) - 2Q_\circ(0.674) + Q_\circ(0.630) - 4Q_\circ(0.5)\\ &+ Q_\circ(0.370) - 2Q_\circ(0.326) + 2Q_\circ(0.206) + Q_\circ(0.091)\big],\end{aligned} \tag{2-40}$$

and

$$\begin{aligned}\xi_4 = \frac{k}{16}\big[&Q_\circ(0.931) + 2Q_\circ(0.841) - 3Q_\circ(0.757) + Q_\circ(0.707) - 6Q_\circ(0.614) + 3Q_\circ(0.544)\\ &- 3Q_\circ(0.456) + 6Q_\circ(0.386) - Q_\circ(0.293) + 3Q_\circ(0.243) - 2Q_\circ(0.159) - Q_\circ(0.069)\big].\end{aligned} \tag{2-41}$$

(3) Using the Gastwirth ($p = 0.3,\ d = 1/3$), the first four LQ-moments for the Pareto distribution will be obtained as follows:

$$\xi_1 = \frac{k}{10}\left[3Q_\circ(0.333) + 4Q_\circ(0.5) + 3Q_\circ(0.667)\right], \tag{2-42}$$

$$\xi_2 = \frac{k}{20}\left[3Q_\circ(0.816) + 4Q_\circ(0.707) + 3Q_\circ(0.577) - 3Q_\circ(0.423) - 4Q_\circ(0.293) - 3Q_\circ(0.184)\right], \tag{2-43}$$

$$\begin{aligned}\xi_3 = \frac{k}{30}\big[&3Q_\circ(0.874) + 4Q_\circ(0.794) + 3Q_\circ(0.693) - 6Q_\circ(0.613) - 8Q_\circ(0.5)\\ &- 6Q_\circ(0.387) + 3Q_\circ(0.307) + 4Q_\circ(0.206) + 3Q_\circ(0.126)\big],\end{aligned} \tag{2-44}$$

and

$$\xi_4 = \frac{k}{40}\left[3Q_{\circ}(0.904) + 4Q_{\circ}(0.841) + 3Q_{\circ}(0.760) - 9Q_{\circ}(0.709) - 12Q_{\circ}(0.614) + 9Q_{\circ}(0.514) - 9Q_{\circ}(0.486) + 12Q_{\circ}(0.386) + 9Q_{\circ}(0.291) - 3Q_{\circ}(0.240) - 4Q_{\circ}(0.159) - 3Q_{\circ}(0.096)\right]. \quad (2\text{-}45)$$

Then, the LQ-skewness and the LQ-kurtosis for each case (median, trimean and Gastwirth) for the Pareto distribution can be obtained by using the results for the first four LQ-moments for the Pareto distribution.

LQ-Moments Estimators:

To estimate the unknown parameters α and k for the Pareto distribution using the LQ-moments, the first and the second sample LQ-moments for the Pareto distribution will be obtained by using the following definition of the rth sample LQ-moments:

$$\hat{\zeta}_r = r^{-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\hat{\tau}_{p,d}(X_{(r-j:r)}), \qquad r = 1,2,\ldots \quad (2\text{-}46)$$

where

$$\hat{\tau}_{p,d}(X_{(r-j:r)}) = p\hat{Q}_{X_{(r-j:r)}}(d) + (1-2p)\hat{Q}_{X_{(r-j:r)}}(1/2) + p\hat{Q}_{X_{(r-j:r)}}(1-d). \quad (2\text{-}47)$$

$\hat{\tau}_{p,d}(X_{(r-j:r)})$ is the quick estimator of the location for the distribution of $X_{(r-j:r)}$ in a random sample of size r, and $\hat{Q}_X(.)$ denotes the linear interpolation estimator of Q(u) given by:

$$\hat{Q}_X(u) = (1-\varepsilon)X_{[n'u]:n} + \varepsilon X_{[n'u]+1:n,} \quad (2\text{-}48)$$

where $\varepsilon = n'u - [n'u]$, $n' = n+1$ and $[n'u]$ denote the integral part of $n'u$. Then, the first two sample LQ-moments will be:

$$\hat{\zeta}_1 = \hat{\tau}_{p,d}(X_{(1:1)}), \quad (2\text{-}49)$$

and

$$\hat{\zeta}_2 = \frac{1}{2}\left[\hat{\tau}_{p,d}(X_{(2:2)}) - \hat{\tau}_{p,d}(X_{(1:2)})\right]. \quad (2\text{-}50)$$

By equating the first two population LQ-moments for the three different cases (median, trimean, and Gastwirth) with the first two sample LQ-moments (2-49) and (2-50) for the Pareto distribution, the LQ-moments estimators for the two unknown parameters will be obtained for each case.

Now, the unknown parameters α and k for the Pareto distribution using the LQ-moments with the median case (LQMEm) will be estimated. Since, the first sample LQ-moments $\hat{\xi}_1$ is a function of α and k and the second sample LQ-moments $\hat{\xi}_2$ is a function also of α and k, then by numerically solving the equations for $\hat{\xi}_1$ and $\hat{\xi}_2$ to obtain the LQ-moments estimates $\hat{\hat{\alpha}}$ and $\hat{\hat{k}}$, then:

$$\hat{\xi}_1 = \hat{\hat{k}}\left[\hat{Q}_{\circ}(0.5)\right], \tag{2-51}$$

and

$$\hat{\xi}_2 = \frac{\hat{\hat{k}}}{2}\left[\hat{Q}_{\circ}(0.707) - \hat{Q}_{\circ}(0.293)\right], \tag{2-52}$$

where

$$\hat{Q}_{\circ}(u) = (1-u)^{-\frac{1}{\hat{\hat{\alpha}}}}. \tag{2-53}$$

For the trimean case the LQ-moments estimates (LQMEt) $\hat{\hat{\alpha}}$ and $\hat{\hat{k}}$ will be obtained by solving the following two equations:

$$\hat{\xi}_1 = \frac{\hat{\hat{k}}}{4}\left[\hat{Q}_{\circ}(0.25) + 2\hat{Q}_{\circ}(0.5) + \hat{Q}_{\circ}(0.75)\right], \tag{2-54}$$

and

$$\hat{\xi}_2 = \frac{\hat{\hat{k}}}{8}\left[\hat{Q}_{\circ}(0.866) + 2\hat{Q}_{\circ}(0.707) - 2\hat{Q}_{\circ}(0.293) - \hat{Q}_{\circ}(0.134)\right], \tag{2-55}$$

and, for the Gastwirth case the LQ-moments estimates (LQMEg) $\hat{\hat{\alpha}}$ and $\hat{\hat{k}}$ will be obtained by solving the following two equations:

$$\hat{\xi}_1 = \frac{\hat{\hat{k}}}{10}\left[3\hat{Q}_{\circ}(0.333) + 4\hat{Q}_{\circ}(0.5) + 3\hat{Q}_{\circ}(0.667)\right], \tag{2-56}$$

and

$$\hat{\xi}_2 = \frac{\hat{\hat{k}}}{20}\left[3\hat{Q}_\circ(0.816) + 4\hat{Q}_\circ(0.707) + 3\hat{Q}_\circ(0.577) - 3\hat{Q}_\circ(0.423) - 4\hat{Q}_\circ(0.293) - 3\hat{Q}_\circ(0.184)\right] \tag{2-57}$$

3 Fuzzy Least-Squares Method

In this section, the fuzzy least-squares method will be used to obtain the fuzzy least-squares estimators for the two-parameter Pareto distribution. Hung and Liu [1] obtained the fuzzy least-squares estimators for the two-parameter Weibull distribution when outliers are present as a robust estimation method. For that purpose, a cluster-wise fuzzy least-squares algorithm with a noise cluster is used. They introduced a numerical comparisons between this fuzzy least-squares algorithm and existing methods (the least-squares, the weighted least-squares, the least absolute deviation and Drapella and Kosznik [2]). According to these comparisons, they suggested that the proposed fuzzy least-squares algorithm is preferable when the sample size is large.

Quandt [9] obtained the estimators for the two-parameter Pareto distribution by using different methods of estimation. Also, he introduced a numerical comparison between these different estimators for the Pareto distribution. He introduced the least-squares estimators for the two-parameter Pareto distribution by using equation (1-1), he found the following:

$$1 - F(x;\alpha,k) = \left(\frac{k}{x}\right)^{\alpha}, \qquad k \le x < \infty,\; k,\alpha > 0 \tag{3-1}$$

Taking logarithms of both sides he obtained:

$$\ln\left(1 - F(x;\alpha,k)\right) = \alpha \ln\left(\frac{k}{x}\right), \qquad k \le x < \infty,\; k,\alpha > 0 \tag{3-2}$$

and

$$\ln\left(1 - F(x;\alpha,k)\right) = \alpha \ln k - \alpha \ln x, \qquad k \le x < \infty,\; k,\alpha > 0 \tag{3-3}$$

Let $x_{(1)} < x_{(2)} < .... < x_{(n)}$ be the order observations in a random sample of size n from $F(x;\alpha,k)$. Then the equation (3-3) gives:

$$\ln\left(1 - F(x_{(i)};\alpha,k)\right) = \alpha \ln k - \alpha \ln x_{(i)}, \quad i = 1,2,...,n \tag{3-4}$$

The parameters α and k may be estimated by least squares from sample estimates of $F(x_{(i)};\alpha,k)$, using, as dependent variable, the logarithm of 1 minus the cumulative distribution of the sample. The least squares estimator of α will be:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n}(t_i - \bar{t})(y_i - \bar{y})}{\sum_{i=1}^{n}(t_i - \bar{t})^2} \tag{3-5}$$

where $t_i = \ln x_{(i)}$, $\bar{t} = \sum_{i=1}^{n} t_i / n$ and $\bar{y} = \sum_{i=1}^{n} y_i / n$.

The corresponding least squares estimator of k may be obtained by substituting into (3-4) the arithmetic mean value of the dependent and independent variable along with the estimator $\hat{\alpha}$ and solving for k. The least square estimator k of will be:

$$\hat{k} = \exp(\bar{t} + \bar{y}/\hat{\alpha}) \tag{3-6}$$

Estimators of the parameters obtained by least squares methods have been shown to be consistent (Quandt [9]).

To obtain the fuzzy least-squares estimators of the two-parameter Pareto distribution, the dependent variable will be the logarithm of 1 minus the cumulative distribution of the sample from equation (3-4). It is common practice of the Pareto plotting technique to use the following estimator for the ordinate of the i^{th} empirical point:

$$y_i = \ln\left(1 - \hat{F}_i\right) \tag{3-7}$$

where $\hat{F}_i$ is a point estimator of $F(x_{(i)}; \alpha, k)$. Many estimators can be used, for example, the mean rank estimator $\hat{F}_i = i/(n+1)$, the median rank estimator $\hat{F}_i = (i - 0.3)/(n + 0.4)$, $\hat{F}_i = (i - 1/2)/n$ and $\hat{F}_i = (i - 3/8)/(n + 1/4)$. In this paper, the mean rank estimator $\hat{F}_i = i/(n+1)$ will be used to represent y_i to obtain the fuzzy least-squares estimators.

Regression analysis is used into the model-fitting of observations. The heterogeneous problem in the regression model is usually difficult to be handled. But the heterogeneity of observations is commonly presented in practice. Hung and Liu [1] first clustered the observations and then use their class memberships as the weights in the weighted least-squares estimation to overcome the heterogeneous problem in the regression model fitting.

Based on this kind of idea, Yang and Ko [10] proposed the cluster-wise fuzzy regression analysis which embeds fuzzy clustering into fuzzy regression model fitting at each step in the iterations. The fuzzy cluster is used to overcome the heterogeneous problem in the fuzzy regression model. Given a data set $\{(x_j, y_j), j = 1,...,n\}$, suppose these observations are heterogeneous and come from c clusters. Of course, if $c = 1$ then the observations are homogeneous. Now, we want to fit a data set to the cluster-wise fuzzy linear regression model:

$$y_j = a_{0i} + a_{1i} x_j, \qquad i = 1,...,c; \quad j = 1,...,n. \tag{3-8}$$

where a_{0i} and $a_{1i} \in R$ are unknown coefficients. Let the membership function $\mu_{ij} \in [0,1]$ with $\sum_{i=1}^{c} \mu_{ij} = 1$ for all $j = 1,\ldots,n$. The notation μ_{ij} is used to represent the membership of the j$^{\text{th}}$ data point (x_j, y_j) belonging to the i$^{\text{th}}$ class. After embedding μ_{ij} to the objective function, one has a cluster-wise objective function:

$$J(\mu, \underline{a_0}, \underline{a_1}) = \sum_{i=1}^{c}\sum_{j=1}^{n} \mu_{ij}^m d^2(a_{0i} + a_{1i}x_j, y_j) = \sum_{i=1}^{c}\sum_{j=1}^{n} \mu_{ij}^m (y_j - a_{0i} - a_{1i}x_j)^2 \tag{3-9}$$

where $\mu = (\mu_{ij})_{c\times n}$, $\underline{a_0} = (a_{01}, a_{02}, \ldots, a_{0c})$, $\underline{a_1} = (a_{11}, a_{12}, \ldots, a_{1c})$ and $m \geq 1$ is the index of fuzziness. Then, the corresponding weighted fuzzy least-squares problem is to minimize the objective function. Now, let $L(\mu, \underline{a_0}, \underline{a_1}, \underline{\lambda})$ be the Lagrangian function with:

$$L(\mu, \underline{a_0}, \underline{a_1}, \underline{\lambda}) = J(\mu, \underline{a_0}, \underline{a_1}) + \sum_{j=1}^{n} \lambda_j (\sum_{i=1}^{c} \mu_{ij} - 1), \tag{3-10}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Set the first derivatives of L with respect to all parameters equal to zero. The following necessary conditions for a minimizer $(\mu, \underline{a_0}, \underline{a_1})$ of J are obtained. That is,

$$a_{1i} = \frac{\sum_{j=1}^{n}\mu_{ij}^m x_j \sum_{j=1}^{n}\mu_{ij}^m y_j - \sum_{j=1}^{n}\mu_{ij}^m x_j y_j \sum_{j=1}^{n}\mu_{ij}^m}{\left(\sum_{j=1}^{n}\mu_{ij}^m x_j\right)^2 - \sum_{j=1}^{n}\mu_{ij}^m \sum_{j=1}^{n}\mu_{ij}^m x_j^2}, \; i = 1,\ldots,c \tag{3-11}$$

$$a_{0i} = \frac{\sum_{j=1}^{n}\mu_{ij}^m y_j - a_{1i}\sum_{j=1}^{n}\mu_{ij}^m x_j}{\sum_{j=1}^{n}\mu_{ij}^m}, \; i = 1,\ldots,c. \tag{3-12}$$

and

$$\mu_{ij} = \left(\sum_{p=1}^{c} \frac{\left(d^2\left(a_{0i} + a_{1i}x_j, y_j\right)\right)^{1/(m-1)}}{\left(d^2\left(a_{0p} + a_{1p}x_j, y_j\right)\right)^{1/(m-1)}}\right)^{-1}, \; i = 1,\ldots,c;, \quad j = 1,\ldots,n. \tag{3-13}$$

Therefore, a cluster-wise FLS algorithm for computing a minimizer of $J(\mu, \underline{a_0}, \underline{a_1})$ has iterations through the necessary conditions (3-11)-(3-13).

A noise cluster is a cluster which contains the noise points or outliers. The concept of a noise cluster proposed by Dave [11] is that all of the points have equal prior opportunity of belonging to a noise cluster. Dave [11] defined a noise prototype as follows: A point *v* is called a noise prototype if the distance $d(x_j, v)$ between the data point x_j and *v* are all equal to a constant δ

, i.e. $d(x_j, v) = \delta$ for $j = 1, \ldots, n$. Hung and Liu [1] applied the noise cluster concept to the cluster-wise FLS. Assuming that the cluster $(c+1)$ is a noise cluster. Then the objective function becomes:

$$J^0(\mu, \underline{a_0}, \underline{a_1}) = \sum_{i=1}^{c+1} \sum_{j=1}^{n} \mu_{ij}^m d_{ij}^2 \text{ with } \sum_{i=1}^{c+1} \mu_{ij} = 1, \quad j = 1, \ldots, n, \tag{3-14}$$

and

$$d_{ij}^2 = \begin{cases} d^2(a_{0i} + a_{1i}x_j, y_j) = (y_j - a_{0i} - a_{1i}x_j)^2, & i = 1, \ldots, c; \quad j = 1, \ldots, n, \\ \delta^2 & , \ i = c+1; \quad j = 1, \ldots, n, \end{cases} \tag{3-15}$$

Where

$$\delta^2 = \gamma \left(\frac{\sum_{i=1}^{c} \sum_{j=1}^{n} d_{ij}^2}{nc} \right), \ \gamma > 0 \tag{3-16}$$

γ is a constant. Thus, when c=1, the algorithm with a noise cluster is iterated with the necessary conditions (3-11) and (3-12) and also with

$$\mu_{ij} = \left(\sum_{p=1}^{c+1} \frac{\left(d_{ij}^2\right)^{1/(m-1)}}{\left(d_{pj}^2\right)^{1/(m-1)}} \right)^{-1}, \ i = 1, \ldots, c+1;, \quad j = 1, \ldots, n. \tag{3-17}$$

Thus, when c=1, the algorithm becomes a robust FLS algorithm for cluster-wise fuzzy regression modal $y_j = a_{01} + a_{11}x_j, \quad j = 1, \ldots, n$. This is because outliers will be dumped to a noise cluster according to the weight of its membership. This algorithm is used to estimate the Pareto parameters as follows. In general, we choose $\gamma = 1$ and the index of fuzziness $m = 2$ (Pal and Bezdek [12] suggested that the best choice for m is 2). Now, let

$$y_i = \ln\left(1 - \hat{F}_i\right), \qquad x_j = \ln t_{(j)}, \quad j = 1, \ldots, n \tag{3-18}$$

In the cluster-wise fuzzy regression model $y_j = a_{01} + a_{11}x_j, \quad j = 1, \ldots, n.$ then the fuzzy least-squares estimates of α and k will be:

$$\hat{\alpha} = -\hat{a}_{11} \text{ and } \hat{k} = \exp\left(-\frac{\hat{a}_{01}}{\hat{a}_{11}} \right) \tag{3-19}$$

Based on these necessary conditions, we can construct the following algorithm to obtain the fuzzy least-squares estimators (FLSEs) for the two-parameter Pareto distribution.

The suggested algorithm to obtain the FLSEs:

(1) –Take the values $\gamma = 1$, $m = 2$ and $c = 1$. Choose an initial $\hat{a}_{01}^{(0)}$, $\hat{a}_{11}^{(0)}$ and $\mu_{1j}^{(0)}$.

(2) – Calculate $\hat{a}_{01}^{(1)}$ and $\hat{a}_{11}^{(1)}$ by using $\mu_{1j}^{(0)}$ and (3-11) and (3-12).

(3) – Calculate $\mu_{1j}^{(1)}$ by using (3-13).

(4) – Compare $\mu_{1j}^{(0)}$ to $\mu_{1j}^{(1)}$, using convenient norm :

$$\left\| \mu_{1j}^{(1)} - \mu_{1j}^{(0)} \right\| = \left(\sum_{j=1}^{n} \left(\mu_{1j}^{(1)} - \mu_{1j}^{(0)} \right)^2 \right)^{\frac{1}{2}} \tag{3-20}$$

If $\mu_{1j}^{(1)}$ is sufficient close to $\mu_{1j}^{(0)}$, i.e. $\left\| \mu_{1j}^{(1)} - \mu_{1j}^{(0)} \right\| \le 10^{-5}$, then stop, otherwise set $\mu_{1j}^{(1)} = \mu_{1j}^{(0)}$, $\hat{a}_{01}^{(1)} = \hat{a}_{01}^{(0)}$, $\hat{a}_{11}^{(1)} = \hat{a}_{11}^{(0)}$ and go to step (2).

4 A Simulation Study of the Pareto Distribution

A simulation study will be introduced to compare between the properties of seven different estimators: fuzzy least square estimators (FLSEs), maximum likelihood estimators (MLEs), L-moment estimators (LMEs), TL-moment estimators (TLMEs) and the three LQ-moment estimators {LQMEm (median), LQMEt (trimean), LQMEg (Gastwirth)} for the unknown parameters of the Pareto distribution. Comparison will be mainly based on their biases and root mean squared errors (RMSEs). The simulation experiments are performed using the Mathcad (14) software, different sample sizes 10, 30, 50 and 100, and different values for the shape parameter $\alpha = 1, 2$ and 3 andfor $k = 3$. For each combination of the sample size and the shape parameters values, the experiment will be repeated 10,000 times. In each experiment, the biases and RMSEs for the estimates of α and k will be obtained and listed in Tables 1 and 2.

5 Results and Conclusion

It is observed in Table 1 that the fuzzy least square estimators (FLSEs) are less unbiased and the minimum RMSEs for all different values of α and for n = 10 and 30 are considered here except for $\alpha = 3$. As far as biases are concerned, the LQMEs are less unbiased for $\alpha = 3$ all times and the minimum RMSEs. For n = 50 and 100, the MLEs are the minimum RMSEs for $\alpha = 2, 3$ with the LQMEs are less unbiased, but for $\alpha = 1$, the FLSEs are less unbiased and the minimum RMSEs for all values of n. Also, it is observed in Table 1 that most of the estimators usually overestimate α all times. The RMSEs of the LQMEs and the MLEs are also quite close to the FLSEs.

Table 1. Biases and RMSEs of the parameter estimators for the MLEs, LMEs TLMEs, LQMEs and FLSEs for different types of moments for α :

n = 10	MLE	LME	TLME	LQMEm	LQMEt	LQMEg	FLSE
$\alpha=1$	1.25639 (1.40917)	1.80390 (1.93480)	1.44614 (1.69085)	1.11681 (1.52290)	0.99923 (1.17339)	1.05596 (1.29459)	0.21639* (0.46131)*
$\alpha=2$	1.69824 (2.04521)	2.07329 (2.39169)	1.77426 (2.29999)	1.35383 (2.21156)	1.07711* (1.50236)	1.21174 (1.76698)	1.14096 (1.41211)*
$\alpha=3$	2.73000 (2.69599)	2.35011 (2.92424)	2.04146 (2.87359)	1.59330 (2.94183)	1.14684* (1.90082)*	1.36914 (2.28805)	2.06971 (2.39664)
n = 30	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha=1$	0.92373 (0.96261)	1.65697 (1.68817)	1.23888 (1.29095)	0.98385 (1.11457)	1.04914 (1.11125)	0.99155 (1.06885)	0.24242* (0.32333)*
$\alpha=2$	1.19235 (1.29152)	1.80554 (1.89369)	1.39823 (1.52438)	1.13217 (1.42957)	1.15157 (1.31376)	1.10507* (1.29780)	1.18680 (1.25968)*
$\alpha=3$	1.40836 (1.58137)	2.35011 (2.92424)	1.55997 (1.78907)	1.28297 (1.77605)	1.25538 (1.53931)*	1.22024* (1.55726)	2.13394 (2.22695)
n = 50	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha=1$	0.86567 (0.88795)	1.63157 (1.64913)	1.21406 (1.24471)	0.94606 (1.02127)	1.04880 (1.08552)	0.98130 (1.02591)	0.25485* (0.30403)*
$\alpha=2$	1.09488 (1.15224)*	1.76661 (1.81790)	1.34945 (1.42408)	1.06907* (1.24609)	1.14764 (1.24283)	1.08621 (1.19951)	1.21974 (1.26756)
$\alpha=3$	1.28374 (1.38782)*	1.89290 (1.99058)	1.47652 (1.61330)	1.19456 (1.49594)	1.25818 (1.42986)	1.19275* (1.39479)	2.17119 (2.22641)
n = 100	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha=1$	0.82606 (0.83677)	1.61737 (1.62603)	1.19815 (1.21404)	0.92212 (0.95965)	1.03750 (1.05606)	0.97259 (0.99493)	0.26809* (0.29261)*
$\alpha=2$	1.03519 (1.06462)*	1.72703 (1.75143)	1.31055 (1.34788)	1.03041* (1.12012)	1.13937 (1.18672)	1.07204 (1.12909)	1.24461 (1.26678)
$\alpha=3$	1.20025 (1.25260)*	1.84174 (1.88968)	1.42982 (1.49723)	1.14120* (1.29692)	1.23236 (1.32158)	1.17313 (1.27611)	2.21198 (2.23919)

The root mean squared errors (RMSEs) are reported in brackets in the table.
**: The least biased value or the least root mean squared errors*

Table 2. Biases and RMSEs of the parameter estimators for the MLEs, LMEs TLMEs, LQMEs and FLSEs for different types of moments for k:

n = 10	MLE	LME	TLME	LQMEm	LQMEt	LQMEg	FLSE
$\alpha = 1$	-1.80115	-1.70038	-1.73400	-1.73694	-1.74983	-1.73703	-0.24561*
	(1.80398)	(1.70775)	(1.74714)	(1.75638)	(1.76045)	(1.74986)	(0.75647)*
$\alpha = 2$	-1.85092	-1.82962	-1.84181	-1.84976	-1.85816	-1.84993	-0.15226*
	(1.85161)	(1.83147)	(1.84518)	(1.85529)	(1.86113)	(1.85344)	(0.40352)*
$\alpha = 3$	-1.86798	-1.86412	-1.87142	-1.87754	-1.88507	-1.87777	-0.09612*
	(1.86828)	(1.86494)	(1.87319)	(1.88021)	(1.88645)	(1.87945)	(0.25977)*
n = 30	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha = 1$	-1.86819	-1.70816	-1.74225	-1.74820	-1.73957	-1.74876	-0.11867*
	(1.86848)	(1.71033)	(1.74684)	(1.75559)	(1.74357)	(1.75317)	(0.41393)*
$\alpha = 2$	-1.88483	-1.83267	-1.84607	-1.85164	-1.85014	-1.85254	-0.05968*
	(1.88491)	(1.83322)	(1.84721)	(1.85373)	(1.85120)	(1.85375)	(0.20841)*
$\alpha = 3$	-1.89032	-1.86412	-1.87341	-1.87720	-1.87585	-1.87809	-0.04524*
	(1.89035)	(1.86494)	(1.87406)	(1.87821)	(1.87636)	(1.87866)	(0.14170)*
n = 50	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha = 1$	-1.88162	-1.70764	-1.74114	-1.75085	-1.74208	-1.74815	-0.09463*
	(1.88172)	(1.70896)	(1.74457)	(1.75522)	(1.74447)	(1.75082)	(0.32712)*
$\alpha = 2$	-1.89133	-1.83241	-1.84607	-1.85221	-1.84877	-1.85142	-0.04702*
	(1.89136)	(1.83274)	(1.84696)	(1.85345)	(1.84941)	(1.85215)	(0.17472)*
$\alpha = 3$	-1.89480	-1.86630	-1.87437	-1.87727	-1.87569	-1.87700	-0.03429*
	(1.89481)	(1.86645)	(1.87489)	(1.87786)	(1.87599)	(1.87735)	(0.10980)*
n = 100	**MLE**	**LME**	**TLME**	**LQMEm**	**LQMEt**	**LQMEg**	**FLSE**
$\alpha = 1$	-1.89119	-1.70800	-1.74046	-1.75427	-1.74465	-1.75012	-0.06268*
	(1.89122)	(1.70866)	(1.74260)	(1.75654)	(1.74587)	(1.75149)	(0.23357)*
$\alpha = 2$	-1.89644	-1.83338	-1.84713	-1.85331	-1.84931	-1.85185	-0.03273*
	(1.83354)	(1.83354)	(1.84782)	(1.85395)	(1.84964)	(1.85222)	(0.11546)*
$\alpha = 3$	-1.89799	-1.86638	-1.87382	-1.87774	-1.87612	-1.87705	-0.02111*
	(1.89799)	(1.86645)	(1.87413)	(1.87805)	(1.87627)	(1.87723)	(0.07626)*

The root mean squared errors (RMSEs) are reported in brackets in the table.
**: The least biased value or the least root mean squared errors*

Comparing all the methods, we conclude that for the parameter α, the FLSEs should be used for estimating α especially for the small sample size. Now consider the estimation of k. In this case, it is observed in Table 2 that most of the estimators usually underestimate k all times. As far as biases are concerned, the FLSEs are less unbiased and the minimum RMSEs for all different values of α and n. Comparing all the methods, we conclude that for the parameter k, the FLSEs should be used for estimating k.

Competing Interests

Authors have declared that no competing interests exist.

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