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On Non-metric Covering Lemmas and Extended Lebesgue-differentiation Theorem

Mangatiana A. Robdera[1*]

[1] *Department of Mathematics, University of Botswana, Private Bag 0022, Gaborone, Botswana.*

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Abstract

We give a non-metric version of the Besicovitch Covering Lemma and an extension of the Lebesgue-Differentiation Theorem into the setting of the integration theory of vector valued functions.

Keywords: Covering lemmas; vector integration; Hardy-Littlewood maximal function; Lebesgue differentiation theorem.

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1 Introduction

The Vitali Covering Lemma that has widespread use in analysis essentially states that given a set $E \subset \mathbb{R}^n$ with Lebesgue measure $\lambda(E) < \infty$ and a cover of E by balls of 'arbitrary small Lebesgue measure', one can find almost cover of E by a finite number of pairwise disjoint balls from the given cover. The more elaborate and more powerful Besicovitch Covering Lemma [1, 2] is known to work for every locally finite Borel measures. Its setting is a finite-dimensional normed space X. For an arbitrary $A \subset X$, and a family of balls $B(a, r_a)$ such that $a \in A$ and $\sup_{a\in A} r_a < \infty$, the theorem states that there exists a constant K depending only on the normed space X such that for some

*Corresponding author: E-mail: robdera@yahoo.com

$m < K$, one can find m disjoint subsets A_i such that for each A_i, the balls $B(a_i, r_{a_i})$ are pairwise disjoint and $\bigcup_{i=1}^{m} \bigcup_{a \in A_i} B(a, r_a)$ still covers A. The Besicovitch Covering Lemma is used to prove the important Lebesgue-differentiation Theorem:

Theorem 1.1. *Let μ be a locally finite Borel measure on $\mathbb{R}^n$. Given $x \in$ suppμ, $f \in L^1(\mathbb{R}^n, \mu)$, then*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu = f(x), \quad \mu - \textit{a.e.}$$

Here, suppμ is $\Omega \setminus U$, where U is the largest open set such that $\mu(U) = 0$ and $B(x, r)$ is the ball centered at x of radius r.

Such a theorem provides an important tool in many areas of analysis, such as, partiall differential equations, harmonic analysis, probability, integral operators, approximation theory, to name just a few. Obviously, such a theorem has several proofs, and extensions in the literature (see e.g. [3, 4, 5, 6, 7]).

The aim of this note is to give a more general, non-metric Besicovitch Covering Lemma that works for any size function (see definition in Section 2) and that will permit to further extend the scope of the Lebesgue Differentiation Theorem to the more general setting of the integral of vector valued functions as introduced in [8] (further developed in [9]).

2 Extended Notion of Integrability

In this section, we recall the main points in the definition of the extended notion of integrability as introduced in [8]. A **size function** is set function $\mu : \Sigma \subset 2^\Omega \to [0, \infty]$ defined on a semiring Σ of subsets of Ω satisfying

- $\mu(\emptyset) = 0$;
- $\mu(A) \le \mu(B)$ whenever $A \subset B$ in Σ (monotonicity)
- $\mu(\bigcup_{n \in \mathbb{N}} A_n) \le \sum_{n \in \mathbb{N}} \mu(A_n)$ for every sequence $n \mapsto A_n$ in Σ such that $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$ (countable subadditivity).

A Σ-*subpartition* P of a subset $A \in 2^\Omega$ is any finite collection $\{I_i; I_i \subset A, I_i \in \Sigma, i = 1, 2 \ldots, n\}$ with the following properties that $\mu(I_i) < \infty$ for all $i \in \{1, \ldots, n\}$, $I_i \subset A$, $I_i \in \Sigma$ and $I_i \cap I_j = \emptyset$ whenever $i \ne j$. A Σ-subpartition $P = \{I_i : i = 1, \ldots, n\}$ is said to be *tagged* if a point $t_i \in I_i$ is chosen for each $i \in \{1, \ldots, n\}$. We write $P := \{(I_i, t_i) : i \in \{1, \ldots, n\}\}$ if we wish to specify the tagging points. We denote by $\Pi(A, \Sigma)$ the collection of all tagged Σ-subpartitions of the set A. The *mesh* or the *norm* of $P \in \Pi(A, \Sigma)$ is defined to be $\|P\| = \max\{\mu(I_i) : I_i \in P\}$.

If $P, Q \in \Pi(A, \Sigma)$, we say that Q is a **refinement** of P and we write $Q \succ P$ if $\|Q\| \le \|P\|$ and $\bigsqcup P \subset \bigsqcup Q$. It is readily seen that the relation $\succ$ is transitive on $\Pi(A, \Sigma)$, and if $P, Q \in \Pi(A, \Sigma)$, then $P \vee Q := \{I \setminus J, I \cap J, J \setminus I : I \in P, J \in Q\} \in \Pi(A, \Sigma)$, $P \vee Q \succ P$ and $P \vee Q \succ Q$. Thus the relation $\succ$ has the upper bound property on $\Pi(A, \Sigma)$. We then infer that the set $\Pi(A, \Sigma)$ is directed by the binary relation $\succ$.

Let $f : \Omega \to V$, where V will denote either a real or a complex normed vector space. Given a Σ-subpartition $P = \{(I_i, t_i) : i \in \{1, \ldots, n\}\} \in \Pi(A, \Sigma)$, we define the (Σ, μ)-*Riemann sum* of f at P to be the vector $f_\mu(P) = \sum_{i=1}^{n} \mu(I_i) f(t_i)$. Thus the function $P \mapsto f_\mu(P)$ is a V-valued net defined on the directed set $(\Pi(A, \Sigma), \succ)$. We thereby say that

Definition 2.1. A function $f : \Omega \to V$ is (Σ, μ)-*integrable* over a set $A \in \Sigma$, with (Σ, μ)-integral $\int_A f d\mu$ if for every $\epsilon > 0$, there exists $P_0 \in \Pi(A, \Sigma)$, such that for every $P \in \Pi(A, \Sigma)$, $P \succ P_0$ we have

$$\left\| \int_A f d\mu - f_\mu(P) \right\| < \epsilon. \tag{2.1}$$

We shall denote by $\mathcal{I}(A, V, \Sigma, \mu)$ the set of all (Σ, μ)-integrable functions over the set A.

It should also be noticed that if the set A is such that $\mu(A) = 0$, then for all subpartitions $P \in \Pi(A)$, $f_\mu(P) = 0$, and thus $\int_A f d\mu = 0$. It follows that

$$\int_A f d\mu = \int_A g d\mu \text{ whenever } \mu\{x \in A : f(x) \neq g(x)\} = 0.$$

We write $f \overset{\mu}{\sim} g$, if $\mu\{x \in A : f(x) \neq g(x)\} = 0$. It is readily seen that the relation $f \overset{\mu}{\sim} g$ is an equivalence relation on $\mathcal{I}(A, V, \Sigma, \mu)$. We shall then denote by $I(A, V, \Sigma, \mu)$ the quotient space $\mathcal{I}(A, V, \Sigma, \mu)/ \overset{\mu}{\sim}$. For $1 \leq p < \infty$, we shall denote by $I^p(\Omega, V, \Sigma, \mu)$ the subspace of $I(\Omega, V, \Sigma, \mu)$ consisting of functions f such that the function $s \mapsto \|f(s)\|_V^p$ is μ-integrable. The space $I^p(\Omega, V, \Sigma, \mu)$ shall be normed by $f \mapsto \|f\|_p = (\int_\Omega \|g\|_V^p \, d\mu)^{\frac{1}{p}}$. The space $I^\infty(\Omega, V, \Sigma, \mu)$ is defined by

$$I^\infty(\Omega, V, \Sigma, \mu) = \left\{f \in \mathfrak{F}(\Omega, V) : \mu\text{-esssup}\|f\|_V < \infty\right\}$$

where $\mathfrak{F}(\Omega, V)$ denotes the set of all V-valued function defined on Ω. The space $I^\infty(\Omega, V, \Sigma, \mu)$ will be normed by $f \mapsto \|f\|_\infty = \mu - \text{esssup}\, \|f\|_V$.

Remark 2.1. We notice that:

- the spaces $\left(I^p(\Omega, V, \Sigma, \mu), \|\cdot\|_p\right)$, $1 \leq p < \infty$, and $\left(I^\infty(\Omega, V, \Sigma, \mu), \|\cdot\|_\infty\right)$ are all Banach spaces whenever V is a Banach space.
- if μ is the Lebesgue measure, then the Lebesgue function space $L^p(\Omega, V)$ is contained in $I^p(\Omega, V, (\Sigma, \mu))$.
- a continuous version of the Dvorestski-Rogers theorem proved in [9, Theorem 16] shows that

$$I^1(\Omega, V, (\Sigma, \mu)) \subsetneq I(\Omega, V, (\Sigma, \mu)).$$

- in the above definition of the integrability, no notion of measurability is required.
- the Hölder's inequality has the following generalization: if $p_1, p_2, \ldots, p_n, r \in [1, \infty)$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = \frac{1}{r}$$

then for every $f_i \in I^{p_i}(\Omega, \mathbb{R}, \Sigma, \mu)$, $i \in \{1, 2, \ldots, n\}$, we have

$$\left[\int_\Omega |f_1 f_2 \cdots f_n|^r \, d\mu\right]^{\frac{1}{r}} \leq \prod_{i=1}^n \left[\int_\Omega |f_i|^{p_i} \, d\mu\right]^{\frac{1}{p_i}}.$$

For more detailed exposition and further results on the notion of extended integral see [10].

3 An Extension of the Besicovitch Covering Lemma

In what follows, (Ω, τ) is a topological vector space and $\Sigma \supset \tau$, and $\mu : \Sigma \subset 2^\Omega \to [0, \infty]$ is a translation invariant size function, that is,

$$\forall A \in \Sigma, \forall \omega \in \Omega, \ \mu(\omega + A) = \mu(A).$$

It is a well-known fact that a topological vector space has a local base consisting of balanced sets. We shall denote by $\mathcal{N}_x^{bal}$ the collection of all balanced neighborhoods of $x \in \Omega$.

We introduce the notion of μ-balls that will take on the role played by metric balls in the standard Besicovitch covering theorem.

Definition 3.1. Let 0 be the zero vector in Ω. We shall call a μ-**ball of size** r **centered at** 0, the set of the form

$$N(0,r) = \bigcap \left\{ N \in \mathcal{N}_0^{bal},\ N \text{ is open, } \mu(N) > r \right\}.$$

For an arbitrary element $\omega \in \Omega$, a μ-**ball of size** r **centered at** ω is defined to be the set

$$N(\omega, r) := \omega + N(0, r).$$

Clearly, $N(\omega, r)$ is balanced, and if $r < r'$, then $N(\omega, r) \subset N(\omega, r')$. By translation invariance, we have $\mu(N(\omega, r)) = \mu(N(0, r))$. More generally, given a subset A of Ω, a set of the form

$$N(A, r) := A + N(0, r)$$

is called a μ**-neighborhood** of the set A. We say that a subset $A \in 2^{\Omega}$ is μ**-bounded** if there exists $r > 0$ such that $A \subset N(0, r)$.

Definition 3.2. We say that the size function μ **satisfies the uniform doubling condition** if there exists $k > 0$ such that for every $\omega \in \Omega$ and for every $r > 0$, we have

$$\mu(N(\omega, r) + N(0, r)) = k\mu(N(0, r)). \tag{3.1}$$

It is easily verified that the Carathéodory extension of the Lebesgue measure is an example of size function that satisfies the uniform doubling condition. In what follows we shall always assume that μ satisfies the uniform doubling condition.

We need some technical lemmas.

Lemma 3.1. *Assume that the size function μ satisfies the uniform doubling condition. Then there exists an integer K_0 such that if $\omega, \omega_1, \ldots, \omega_{K_0} \in \Omega$ with the properties that $N(\omega, 1) \cap N_i(\omega_i, 1) \neq \emptyset$ for each i, then some $N_i(\omega_i, 1)$ contains ω_j for some $i \neq j$.*

Proof. Let $\omega_1, \ldots, \omega_m \in \Omega$ be such that $N(\omega, 1) \cap N(\omega_i, 1) \neq \emptyset$ for each $i \in \{1, \ldots, m\}$, and no ω_i belongs to $N(\omega_j, 1)$ for $i \neq j$. Then we notice that $N(\omega_i, 1) \subset N(\omega, 1) + N(0, 1)$ for each i. Indeed, if $x \in N(\omega_i, 1)$, then $x - \omega_0 \in N(0, 1)$ where $\omega_0 \in N(\omega, 1) \cap N(\omega_i, 1)$. On the other hand, $\omega_0 - \omega \in N(0, 1)$, and so

$$x - \omega = (x - \omega_0) + (\omega_0 - \omega) \in N(0, 1) + N(0, 1)$$

which implies that

$$x \in \omega + N(0, 1) + N(0, 1) = N(\omega, 1) + N(0, 1).$$

It then follows that

$$\bigcup_{i=1}^{m} N(\omega_i, 1) \subset N(\omega, 1) + N(0, 1).$$

We next claim that $N(\omega_i, \frac{1}{2k})$ are disjoint μ-balls, that are obviously contained in $\bigcup_{i=1}^{m} N(\omega_i, 1)$.

Indeed, if $N(\omega_i, \frac{1}{2k}) \cap N(\omega_j, \frac{1}{2k}) \neq \emptyset$, then

$$N(\omega_i, \frac{1}{2k}) \cup N(\omega_j, \frac{1}{2k}) \subset N(\omega_i, \frac{1}{2k}) + N(0, \frac{1}{2k}) \tag{3.2}$$

It follows from (3.1) that

$$N(\omega_i, \frac{1}{2k}) + N(0, \frac{1}{2k}) \subset N(\omega_i, \frac{1}{2}) \subset N(\omega_i, 1). \tag{3.3}$$

Combining (3.2) and (3.3) would imply that $\omega_j \in N(\omega_i, 1)$. This contradiction proves our claim.

It then follows again from (3.1) that

$$\begin{aligned}\frac{m}{2k} &= \sum_{i=1}^{m}\mu\left(N(\omega_i,1/2k)\right)=\mu\left(\bigcup_{i=1}^{m}N(\omega_i,1/2k)\right)\\ &\le \mu\left(\bigcup_{i=1}^{m}N(\omega_i,1)\right)\le\mu\left(N(\omega,1)+N(0,1)\right)\\ &= k\mu(N(0,1))=k.\end{aligned}$$

Hence $m \le 2k^2$. The proof is complete. $\square$

With a few obvious technical changes, the following variant of the above lemma can safely be established.

Lemma 3.2. *There exists an integer K_1 such that if $\omega,\omega_1,\ldots,\omega_{K_1}\in\Omega$ with the properties that $N(\omega,r)\cap N(\omega_i,r_i)\neq\emptyset$ for each $i,$ where $r_i\ge\frac{2}{3}r>0,$ then some $N_i(\omega_i,r_i)$ contains ω_j for some $i\neq j.$*

We now state and prove our first extension of the Besicovitch covering lemma. The proof follows the same transcendental induction argument as in the proof of the corresponding standard metric case (see e.g. [11]). One simply replaces closed balls with μ-balls.

Theorem 3.3. *Let $\mathcal{F}$ be a collection of μ-balls of size at most $R,$ for some $R>0.$ Let C be the set of the centers of the μ-balls in $\mathcal{F}.$ Then there exists an integer $K>0,$ such that for each $1\le k<K,$ there exists a countable collection $\mathcal{C}_k\subset\mathcal{F}$ such that*

$$C\subset\bigcup_{k=1}^{K}\bigcup_{N\in\mathcal{C}_k}N.$$

Proof. Let $R_0:=\sup\{\mu(N):N\in\mathcal{F}\}$. Pick $N(\omega,r)\in\mathcal{F}$ such that $r\ge\frac{9}{10}R_0$. We consider the integer K_1 given by Lemma 3.2. Define

$$\mathcal{F}_1=\{N(\omega,r)\},\mathcal{F}_2=\mathcal{F}_3=\ldots=\mathcal{F}_K=\emptyset$$

where $K=K_1+1$. We let $\mathcal{T}=\{N(\omega',r)\in\mathcal{F}:\ \omega'\in N(\omega,r)\}$ and $\mathcal{R}=\mathcal{F}\setminus\mathcal{T}\cup\mathcal{F}_1\cup\cdots\cup\mathcal{F}_K$. Let $\mathcal{H}$ be the collection of partitions of $\mathcal{F}$ of the form $\{\mathcal{F}_1,\cdots,\mathcal{F}_K,\mathcal{T},\mathcal{R}\}$ with the following properties:

1. Each $\mathcal{F}_j$ consists of disjoint μ-balls (from $\mathcal{F}$).
2. If $N(\omega,r)\in\mathcal{F}_j$, and if $N(\omega',r')\in\mathcal{F}$ is such that $\omega'\in N(\omega,\frac{9}{10}r)$, then $N(\omega',r')\in\mathcal{T}$.
3. If $N(\omega,r)\in\mathcal{F}_j$, and $N(\omega',r')\in\mathcal{R}$, then $r\ge\frac{9}{10}r'$.
4. If $N(\omega,r)\in\mathcal{T}$, then some $N(\omega',r')\in\mathcal{F}_\infty\cup\cdots\cup\mathcal{F}_\mathcal{K}$.

The step above shows that $\mathcal{H}\neq\emptyset$.
We partially order $\mathcal{H}$ as follows

$$\begin{gathered}\{\mathcal{F}_1,\cdots,\mathcal{F}_K,\mathcal{T},\mathcal{R}\}\quad\prec\quad\{\mathcal{F}_1',\cdots,\mathcal{F}_K',\mathcal{T}',\mathcal{R}'\}\\ \Longleftrightarrow\\ \mathcal{F}_i\subset\mathcal{F}_i',\ \forall i,\quad\mathcal{T}\subset\mathcal{T}',\quad\mathcal{R}\supset\mathcal{R}'.\end{gathered}$$

We are done if we show that there exists $\{\mathcal{F}_1,\cdots,\mathcal{F}_K,\mathcal{T},\mathcal{R}\}\in\mathcal{H}$ such that $\mathcal{R}=\emptyset$.

Let $\{\mathcal{F}_1^\alpha,\cdots,\mathcal{F}_K^\alpha,\mathcal{T}^\alpha,\mathcal{R}^\alpha\}_{\alpha\in A}$ be a totally ordered family in $\mathcal{H}$. Then clearly,

$$\left\{\bigcup_{\alpha\in\mathcal{A}}\mathcal{F}_1^\alpha,\cdots,\bigcup_{\alpha\in\mathcal{A}}\mathcal{F}_K^\alpha,\bigcup_{\alpha\in A}\mathcal{T}^\alpha,\bigcap_{\alpha\in A}\mathcal{R}^\alpha\right\}$$

is an upper bound. Hence by the Zorn's Lemma, $\mathcal{H}$ has a maximal element, say

$$\mathcal{M} = \{\mathcal{F}_1, \cdots, \mathcal{F}_K, \mathcal{T}, \mathcal{R}\}.$$

We shall show that $\mathcal{R} = \emptyset$. Assume that $\mathcal{R} \neq \emptyset$. Let $R_1 := \sup\{\mu(N) : N \in \mathcal{R}\}$. Pick $N(\omega, r) \in \mathcal{R}$ such that $r \geq \frac{9}{10}R_1$. Then $N(\omega, r)$ is disjoint from all the balls of at least one of the $\mathcal{F}_i$, because otherwise we would have a contradiction with Lemma 3.2. Let k_1 be the smallest element in $\{1, \ldots, K\}$ with such a disjointness property. Consider

$$\mathcal{M}^* := \{\mathcal{F}_1, \cdots, \mathcal{F}_{k_1} \cup \{N(\omega, r)\}, \mathcal{F}_{k_1+1}, \ldots, \mathcal{T}^*, \mathcal{R}^*\}$$

where $\mathcal{T}^*$are those balls in $\mathcal{R}$ whose centers are in $N(\omega, r)$, and $\mathcal{R}^*$denotes the reminder. Then $\mathcal{M}^* \in \mathcal{H}$ and $\mathcal{M} \prec \mathcal{M}^*$. This contradicts the maximality of $\mathcal{M}$, and hence finishes our proof.

□

We now introduce the definition of a non-metric Besicovitch covering.

Definition 3.3. Let $\mathcal{F}$ be a collection of non trivial μ-balls in Ω. We say that $\mathcal{F}$ is a **μ-fine Besicovitch covering** for a set $A \in 2^{\Omega}$ if for every $a \in A$ and every $\epsilon > 0$, there exists a μ-ball $N(a, r) \in \mathcal{F}$ such that $r < \epsilon$.

Our next result generalizes the Besicovitch measure-theoretical covering Lemma.

Theorem 3.4. *Let $A \subset \Omega$, $\mu(A) < \infty$, and let $\mathcal{F}$ be a μ-fine Besicovitch covering of A. Then there exists a countable subcollection $\mathcal{C}$ of $\mathcal{F}$, consisting of disjoint μ-balls such that*

$$\mu(A \setminus \bigcup_{N \in \mathcal{C}} N) = 0. \tag{3.4}$$

Proof. We assume that $\mu(A) > 0$, otherwise the statement is trivial. Since $\mu(A) < \infty$, we also can assume without loss of generality that the μ-balls elements of $\mathcal{F}$ are all of size at most 1. Then by Lemma 3.3, there exists an integer $K > 0$, such that for each $1 \leq k < K$, there exists a countable collection $\mathcal{C}_k \subset \mathcal{F}$ such that $A \subset \bigcup_{k=1}^{K} \bigcup_{N \in \mathcal{C}_k} N$. Hence there exist $k \in \{1, \ldots, K\}$, and disjoint μ-balls in $\mathcal{F}$ centered at $\omega_1, \ldots, \omega_{L_1} \in \Omega$ such that

$$\mu(A \cap \bigcup_{i=1}^{L_1} N(\omega_i, r_i)) \geq \frac{\mu(A)}{K+1}.$$

It follows that

$$\mu(A \setminus \bigcup_{i=1}^{L_1} N(\omega_i, r_i)) \leq (1 - \frac{1}{K+1})\mu(A).$$

Next, we let $A_2 = A \setminus \bigcup_{i=1}^{L_1} N_i$. If $\mu(A_2) = 0$, the process terminates and the theorem is proven. Otherwise we let

$$\mathcal{F}_2 = \{N \in \mathcal{F} : \text{ center of } N \text{ in } A_2,\ N \cap N(\omega_i, r_i) = \emptyset,\ i = 1, \ldots, L_1\}$$

and we then apply the same argument as above to the pair $(A_2, \mathcal{F}_2)$ to obtain disjoints μ-balls centered at $\omega_{L_1+1}, \ldots, \omega_{L_2} \in \Omega$ such that

$$\mu(A \setminus \bigcup_{i=1}^{L_2} N(\omega_i, r_i)) \leq (1 - \frac{1}{K+1})^2\mu(A).$$

Repeating the process m times, we will obtain a collection of L_m μ-balls centered at $\omega_{L_{m-1}+1}, \ldots, \omega_{L_m} \in \Omega$ such that

$$\mu(A \setminus \bigcup_{i=1}^{m} N(\omega_i, r_i)) \leq (1 - \frac{1}{K+1})^m\mu(A). \tag{3.5}$$

If for some $m \in \mathbb{N}$,

$$\mu(A \setminus \bigcup_{i=1}^{m} N(\omega_i, r_i)) = 0$$

the process terminates and the theorem is proven. Otherwise, (3.5) holds for all $m \in \mathbb{N}$. Letting $m \to \infty$, the claim follows.

□

4 Maximal Function

In this section we extend the Lebesgue-Differentiation Theorem to the setting of functions in the space $I^1(\Omega, X, \mu)$ where Ω is a topological vector space, X is a normed vector space and μ is a translation invariant size function that satisfies the uniform doubling condition.

Definition 4.1. Let $f \in I^1(\Omega, X, \mu)$. We define the *Hardy-Littlewood Maximal Function* $M_\mu f$ by

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(N(x,r))} \int_{N(x,r)} \|f(\omega)\| \, d\mu(\omega).$$

Further, and more generally, if $\nu : \Sigma \to X$ is an additive vector measure of bounded variation, then we define

$$M_\mu \nu(x) = \sup_{r>0} \frac{\|\nu(N(x,r))\|}{\mu(N(x,r))}.$$

Recall that the variation of an additive vector measure $\nu : \Sigma \to X$ is defined to be

$$\|\nu\|_1 = \sup_{\pi} \sum_{A \in \pi} \|\nu(A)\|$$

where the supremum is taken over all Σ-partitions π of Ω.

The principal result about maximal function is the following:

Proposition 4.1. *Let μ be a translation invariant size function that satisfies the uniform doubling condition. If $\nu : \Sigma \to X$ is an additive vector measure and $\alpha > 0$, then*

$$\mu\left(\{x \in \Omega : M_\mu \nu(x) > \alpha\}\right) \leq \frac{1}{\alpha} \|\nu\|_1 . \tag{4.1}$$

Here and hereforth $\{M_\mu \nu > \alpha\}$ is a short for $\{x \in \Omega : M_\mu \nu f(x) > \alpha\}$.

Proof. Let $A = \{M_\mu \nu > \alpha\}$, and let $\mathcal{F}$ consist of all μ-balls $N(x, r_x)$ of size at most $r_x \leq 1$, centered at points x of A such that

$$\frac{\|\nu(N(x,r))\|}{\mu(N(x,r))} > \alpha.$$

If $A = \emptyset$, there is nothing to prove. If $A \neq \emptyset$, the hypotheses of Theorem 3.4 are satisfied, and it follows that there exists a family of disjoint μ-balls $\{N(x_i, r_i) \in \mathcal{F} : i \in \mathbb{N}\}$ such that $\mu(A \setminus \bigcup_{i=1}^{\infty} N(x_i, r_i)) = 0$. Hence

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(N_i) \leq \frac{1}{\alpha} \sum_{i=1}^{\infty} \|\nu(N(x,r))\| \leq \frac{1}{\alpha} \|\nu\|_1 .$$

The proof is complete. □

Since every $f \in I^1(\Omega, X, \mu)$ naturally defines an additive set function of bounded variation given by

$$A \mapsto \nu(A) = \int_A f d\mu,$$

the following result is a particular case of the above proposition.

Proposition 4.2. *Let μ be a translation invariant size function that satisfies the uniform doubling condition. If $f \in I^1(\Omega, X, \mu)$ and $\alpha > 0$, then*

$$\mu\left(\{x \in \Omega : M_\mu f(x) > \alpha\}\right) \leq \frac{1}{\alpha} \|f\|_1 . \tag{4.2}$$

We now prove the Lebesgue-Differentiation Theorem.

Theorem 4.1. *Let Ω be a locally compact topological vector space. Let $\mu : \Sigma \subset 2^\Omega \to [0, \infty]$ is a finitely additive size function. If $f \in I^1(\Omega, X, \mu)$, then*

$$\lim_{r \to 0} \frac{1}{\mu(N(0,r))} \int_{N(0,r)} f d\mu = f(x), \quad \mu\text{-a.e.}$$

Proof. By local compactness of Ω, it is quickly seen that $f \in I^1(\Omega, X, \mu)$ if and only if $1_K f \in I^1(\Omega, X, \mu)$ for every K compact subset of Ω. We then may assume without loss of generality that Ω is compact. Given $\epsilon > 0$, let g be continuous on Ω such that $\|f - g\|_1 < \epsilon$. Since we have for every $x \in \Omega$,

$$\lim_{r \to 0} \frac{1}{\mu(N(0,r))} \int_{N(0,r)} g d\mu = g(x)$$

we have for every $\alpha > 0$,

$$\begin{aligned} E_\alpha &:= \left\{ x \in \Omega : \limsup_{r \to 0} \left\| \frac{1}{\mu(N(0,r)} \int_{N(0,r)} f d\mu - f(x) \right\| > \alpha \right\} \\ &= \left\{ x \in \Omega : \limsup_{r \to 0} \left\| \frac{1}{\mu(N(0,r))} \int_{N(0,r)} (f-g) d\mu - (f-g)(x) \right\| > \alpha \right\}. \end{aligned}$$

We are done if we show that $\mu(E_\alpha) = 0$. We notice that

$$\limsup_{r \to 0} \left\| \frac{1}{\mu(N(0,r))} \int_{N(0,r)} (f-g) d\mu - (f-g)(x) \right\| < M_\mu(f-g) + \|f - g\| .$$

It follows that

$$E_\alpha \subset \left\{ x \in \Omega : M_\mu(f-g) > \frac{\alpha}{2} \right\} \cup \left\{ x \in \Omega : \|f(x) - g(x)\| > \frac{\alpha}{2} \right\}.$$

By Proposition 4.1 and the Tchebytchev's inequality, it follows that

$$\mu(E_\alpha) \leq \frac{4}{\alpha} \|f - g\|_1 < \frac{4\epsilon}{\alpha}$$

for all $\epsilon > 0$. Hence $\mu(E_\alpha) = 0$. The proof is complete. □

5 Conclusion

We have given a treatment of the Lebesgue-Differentiation Theorem in the setting of the integration theory of vector valued functions. The treatment leans heavily of the machinery developped in the first half of the paper which provides a non-metric version of the Besicovitch covering lemma. The author believes that the interest in such generalizations lies not only in the fact that we have more general results, but also in the light they shed on the classical situation.

Competing Interests

Author declares that no competing interests exist.

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