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I-pure Submodules, I-FP-injective Modules and I-flat Modules

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Abstract

Let R be a ring and I an ideal of R . We define and study I-pure submodules , I-FP-injective modules, I-flat modules , I-coherent rings and I-semihereditary rings. Using the concepts of I-FP-injectivity and I-flatness of modules, we also present some characterizations of I-coherent rings and I-semihereditary rings.

Keywords: I-pure submodules; I-FP-injective modules; I-flat modules; I-coherent rings; I-semi-hereditary rings.

2010 Mathematics subject classification: 16D40, 16D50, 16P70, 16E60.

1 Introduction

Throughout this paper, m, n are positive integers, R is an associative ring with identity, I is an ideal of R, $J = J(R)$ is the Jacobson radical of R and all modules considered are unitary. For any module M, M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Z}$ is the set of integers. In general, for a set S, we write $S^{m\times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of S, and S_n (resp., S^n) for the set of all formal $n \times 1$ (resp., $1 \times n$) matrices

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whose entries are elements of S. Let N be a left R-module, $X \subseteq N_n$ and $A \subseteq R^n$. Then we definite $\mathbf{r}_{N_n}(A) = \{u \in N_n : au = 0, \forall a \in A\}$, and $\mathbf{l}_{R^n}(X) = \{a \in R^n : ax = 0, \forall x \in X\}$.

Recall that a left R-module M is called ***FP-injective*** [1] or ***absolutely pure*** [2] if $\text{Ext}^1_R(A, M) = 0$ for every finitely presented left R-module A; a right R-module M is flat if and only if $\text{Tor}^R_1(M, A) = 0$ for every finitely presented left R-module A; a ring R is left coherent [3] if every finitely generated left ideal of R is finitely presented, or equivalently, if every finitely generated submodule of a projective left R-module is finitely presented ; a ring R is left semihereditary [4] if every finitely generated left ideal of R is projective, or equivalently, if every finitely generated submodule of a projective left R-module is projective. We recall also that: given a right R-module U with submodule U', then U' is called a *pure submodule* of U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism for every finitely presented left R-module V. Pure submodules, FP-injective modules, flat modules, coherent rings, semihereditary rings , and their generalizations have been studied extensively by many authors (see, for example, [1, 3, 5, 6, 7, 8]).

In this article, we wish to introduce a new generalization for pure submodules, FP-injective modules, flat modules, coherent rings, semihereditary rings respectively.

Let I be an ideal of R. In section 2 of this paper, we introduce the concept of *I-pure submodules.* Given a right R-module U with submodule U', then U' is called an I-pure submodule of U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism for every I-finitely presented left R-module V, where a left R-module V is said to be I-finitely presented, if there is a positive integer m and an exact sequence of left R-modules $0 \to K \to R^m \to V \to 0$ with K a finitely generated submodule of I^m. We give some characterizations and properties of I-pure submodules.

In section 3 and section 4, we introduce the concepts of *I-FP-injective modules* and *I-flat modules*. A left R-module M is called I-FP-injective, if $\text{Ext}^1_R(V, M) = 0$ for every I-finitely presented left R-module V; a right R-module M is called I-flat, if $\text{Tor}^R_1(M, V) = 0$ for every I-finitely presented left R-module V. We give some characterizations and properties of I-FP-injective modules and I-flat modules. For instance, we prove that a left R-module M is I-FP-injective if and only if it is I-pure in every module containing it.

In section 5, we introduce the concepst of *I-coherent rings* and *I-semihereditary rings*. The ring R is called I-coherent if every finitely generated left ideal in I is finitely presented. The ring R is called I-semihereditary if every finitely generated left ideal in I is projective. We give some characterizations and properties of I-coherent rings and I-semihereditary rings, especially, I-coherent rings and I-semihereditary rings are characterized by I-FP-injective modules and I-flat modules, some interesting results are obtained. For instance, we prove that R is a left I-coherent ring $\Leftrightarrow$ any direct product of I-flat right R-modules is I-flat $\Leftrightarrow$ any direct limit of I-FP-injective left R-modules is I-FP-injective $\Leftrightarrow$ every right R-module has an I-flat preenvelope; R is a left I-semihereditary ring $\Leftrightarrow$ R is left I-coherent and every submodule of an I-flat right R-module is I-flat $\Leftrightarrow$ every quotient module of an I-FP-injective left R-module is I-FP-injective $\Leftrightarrow$ every left R-module has a monic I-FP-injective cover $\Leftrightarrow$ every right R-module has an epic I-flat envelope.

2 I-pure Submodules

Recall that a left R-module V is said to be *(m,n)-presented* [8], if there is an exact sequence of left R-modules $0 \to K \to R^m \to V \to 0$ with K n-generated. We extend the definitions of (m, n)-presented modules and finitely presented modules respectively as follows.

Definition 2.1. *A left R-module V is said to be I-(m,n)-presented, if there is an exact sequence of left R-modules $0 \to K \to R^m \to V \to 0$ with K an n-generated submodule of I^m. A left R-module V is said to be I-finitely presented if it is I-(m,n)-presented for a pair of positive integers m, n.*

Clearly, a left R-module V is (m, n)-*presented* if and only if it is R-(m, n)-presented, a left R-module V is finitely presented if and only if it is R-finitely presented.

Definition 2.2. *Given a right R-module U with submodule U'. Then:*

(1) *U' is called I-(m,n)-pure in U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism for every I-(m,n)-presented left R-module V. U' is said to be I-$(m, \infty$-pure (resp., I-(∞, n)-pure in U in case U' is I-(m,n)-pure in U for all positive integers n (resp., m).*

(2) *U' is called I-pure in U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism for every I- finitely presented left R-module V.*

Example 2.3. (1) *It is easy to see that U' is (m,n)-pure in U if and only if U' is R-(m,n)-pure in U. U' is pure in U if and only if U' is R-pure in U.*

(2) *Let I_1 and I_2 be two ideals with $I_1 \subseteq I_2$. If U' is I_2-(m,n)-pure in U, then U' is I_1-(m,n)-pure in U.* □

Theorem 2.4 *Let $U'_R \leq U_R$. Then the following statements are equivalent*:

(1) *U' is I-(m, n)-pure in U.*

(1)$'$ *For all $C \in I^{n\times m}$, the canonical map $U' \otimes_R (R^m/R^nC) \to U \otimes_R (R^m/R^nC)$ is a monomorphism.*

(2) *For every I-(m, n)-presented left R-module V, the canonical map $\mathrm{Tor}_1^R(U, V) \to \mathrm{Tor}_1^R(U/U', V)$ is surjective.*

(3) *For all $C \in I^{n\times m}$, $(U')^m \cap U^nC = (U')^nC$.*

(4) *For every n-generated submodule T of ${}_RI^m$, $(U')^m \cap UT = U'T$.*

(5) *For every I-(n, m)-presented right R-module A, the canonical map $\mathrm{Hom}_R(A, U) \to \mathrm{Hom}_R(A, U/U')$ is surjective.*

(5)$'$ *For all $C \in I^{n\times m}$, the canonical map*

$$\mathrm{Hom}_R(R_n/CR_m, U) \to \mathrm{Hom}_R(R_n/CR_m, U/U')$$

is surjective.

(6) *For every I-(n, m)-presented right R-module A, the canonical map $\mathrm{Ext}^1(A, U') \to \mathrm{Ext}^1(A, U)$ is a monomorphism.*

Proof. (1)⇔(1)$'$ and (5)⇔(5)$'$ are obvious.

(1)⇔(2). This follows from the exact sequence

$$\mathrm{Tor}_1^R(U, V) \to \mathrm{Tor}_1^R(U/U', V) \to U' \otimes V \to U \otimes V.$$

(1)⇒(3). Let $C = (c_{ij})_{n\times m} \in I^{n\times m}$ and $x \in (U')^m \cap U^nC$. Then there exist $a_1, a_2, \cdots, a_m \in U'$, $u_1, u_2, \cdots, u_n \in U$ such that $x = (a_1, a_2, \cdots, a_m)$ and $a_i = \sum_{j=1}^{n} u_jc_{ji}$, $i = 1, 2, \cdots, m$. Let $V = R^m/L$, where

$$L = R\alpha_1 + \cdots + R\alpha_n, \alpha_j = (c_{j1}, c_{j2}, \cdots, c_{jm}), j = 1, 2, \cdots, n$$

. Then V is I-(m, n)-presented and we have $\sum_{i=1}^{m} a_i \otimes \overline{e_i} = \sum_{i=1}^{m}(\sum_{j=1}^{n} u_jc_{ji}) \otimes \overline{e_i} = \sum_{j=1}^{n}(u_j \otimes \sum_{i=1}^{m} c_{ji}\overline{e_i}) = \sum_{j=1}^{n}(u_j \otimes \overline{\alpha_j}) = 0$ in $U \otimes V$. Since U' is I-(m, n)-pure in U, $\sum_{i=1}^{m} a_i \otimes \overline{e_i} = 0$ in $U' \otimes V$. So from the exactness of the sequence $U' \otimes L \overset{1_{U'}\otimes\iota}{\to} U' \otimes R^m \overset{1_{U'}\otimes\pi}{\to} U' \otimes V \to 0$, we have $\sum_{i=1}^{m} a_i \otimes e_i = (1_{U'} \otimes \iota)(\sum_{j=1}^{n} u'_j \otimes \alpha_j) = \sum_{j=1}^{n} u'_j \otimes \alpha_j = \sum_{j=1}^{n} u'_j \otimes (\sum_{i=1}^{m} c_{ji}e_i) = \sum_{i=1}^{m}(\sum_{j=1}^{n} u'_jc_{ji}) \otimes e_i$ for some $u'_1, u'_2, \cdots, u'_m \in U'$. This follows that $a_i = \sum_{j=1}^{n} u'_jc_{ji}$, $i = 1, 2, \cdots, m$, thus $x \in (U')^nC$. But $(U')^nC \subseteq (U')^m \cap U^nC$, so $(U')^m \cap U^nC = (U')^nC$.

(3)⇒(4). Let $T = Rb_1 + \cdots + Rb_n$, where $b_j = (c_{1j}, c_{2j}, \cdots, c_{mj}) \in I^m$, $j = 1, 2, \cdots, n$. If $x = (a_1, \cdots, a_m) = \sum_{j=1}^{n} u_jb_j \in (U')^m \cap UT$, where each $a_i \in U'$ and each $u_j \in U$, then

$x = (u_1, u_2, \cdots, u_n)C \in U^nC \cap (U')^m$, where C is the $n \times m$ matrix with row vectors $b_1, \cdots, b_n$. Clearly, $C \in I^{n\times m}$. By (3), $x = (u_1', u_2', \cdots, u_n')C$ for some $u_1', u_2', \cdots, u_n' \in U'$. It follows that $x \in U'T$, and so $(U')^m \cap UT = U'T$.

(4)⇒(5). Consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_K} & R^n & \xrightarrow{\pi_2} & A & \longrightarrow & 0 \\ & & & & & & \big\downarrow f & & \\ 0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' & \longrightarrow & 0 \end{array}$$

where $f \in \mathrm{Hom}_R(A, U/U')$ and K is an m-generated submodule of I^n, with generators $y_i = (c_{i1}, c_{i2}, \cdots, c_{in})$, $i = 1, 2, \cdots, m$. Since R^n is projective, there exist $g \in \mathrm{Hom}_R(R^n, U)$ and $h \in \mathrm{Hom}_R(K, U')$ such that the diagram commutes. Now let $b_j = (c_{1j}, c_{2j}, \cdots, c_{mj}) \in I^m$, $j = 1, 2, \cdots, n$, $T = Rb_1 + \cdots + Rb_n$ and $u_i = \sum_{j=1}^n g(e_j)c_{ij}$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ (with 1 in the jth position and 0's in all other positions), $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Then $u_i = g(\sum_{j=1}^n e_jc_{ij}) = g(y_i) = h(y_i) \in U'$, $i = 1, 2, \cdots, m$. Note that $(u_1, u_2, \cdots, u_m) = \sum_{j=1}^n g(e_j)b_j \in UT$, by (4), $(u_1, u_2, \cdots, u_m) = \sum_{j=1}^n u_j'b_j$ for some $u_1', u_2', \cdots, u_n' \in U'$. Therefore, $u_i = \sum_{j=1}^n u_j'c_{ij}$, $i = 1, 2, \cdots, m$. Define $\sigma \in \mathrm{Hom}_R(R^n, U')$ such that $\sigma(e_j) = u_j'$, $j = 1, 2, \cdots, n$. Then $\sigma i_K = h$. Finally, we define $\tau : A \to U$ by $\tau(z + K) = g(z) - \sigma(z)$, then τ is a well-defined right R-homomorphism and $\pi_1\tau = f$. Whence $\mathrm{Hom}_R(A, U) \to \mathrm{Hom}_R(A, U/U')$ is surjective.

(5)⇒(3). Suppose that $C = (c_{ij})_{n\times m} \in I^{n\times m}$ and $x \in (U')^m \cap U^nC$. Then $x = (a_1, a_2, \cdots, a_m) = (u_1, u_2, \cdots, u_n)C$ for some $a_1, a_2, \cdots, a_m \in U'$ and $u_1, u_2, \cdots, u_n \in U$. Take $y_i = (c_{1i}, c_{2i}, \cdots, c_{ni})$ $(i = 1, 2, \cdots, m)$, $K = y_1R + y_2R + \cdots + y_mR$ and $A = R^n/K$. Then A is I-(n, m)-presented and we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_K} & R^n & \xrightarrow{\pi_2} & A & \longrightarrow & 0 \\ & & \big\downarrow f_1 & & \big\downarrow f_2 & & & & \\ 0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' & \longrightarrow & 0 \end{array}$$

where f_2 is defined by $f_2(e_j) = u_j$, $j = 1, 2, \cdots, n$ and $f_1 = f_2|_K$. Define $f_3 : A \to U/U'$ by $f_3(z + K) = \pi_1 f_2(z)$. Then it is easy to see that f_3 is well defined and $f_3\pi_2 = \pi_1 f_2$. By hypothesis, $f_3 = \pi_1\tau$ for some $\tau \in \mathrm{Hom}_R(A, U)$. Now we define $\sigma : R^n \to U'$ by $\sigma(z) = f_2(z) - \tau\pi_2(z)$. Then $\sigma \in \mathrm{Hom}_R(R^n, U')$ and $i_{U'}\sigma = f_2$. Hence $a_i = f_2(y_i) = \sigma(y_i) = \sum_{j=1}^n \sigma(e_j)c_{ji}$, $i = 1, 2, \cdots, m$, and $x = (\sigma(e_1), \sigma(e_2), \cdots, \sigma(e_n))C \in (U')^nC$. Therefore $(U')^m \cap U^nC = (U')^nC$.

(3)⇒(1). Let ${}_RV$ be I-(m, n)-presented. Without loss of generality, write $V = R^m/L$, where

$$L = R\alpha_1 + \cdots + R\alpha_n,\ \alpha_j = (c_{j1}, c_{j2}, \cdots, c_{jm}) \in I^m, j = 1, 2, \cdots, n.$$

If $\sum_{k=1}^s a_k \otimes b_k = 0$ in $U \otimes V$, where $a_k \in U'$, $b_k = \sum_{j=1}^m r_{kj}\overline{e_j} \in V$, then $\sum_{j=1}^m(\sum_{k=1}^s a_kr_{kj}) \otimes \overline{e_j} = 0$ in $U \otimes V$. Consider the exact sequence of $U \otimes L \overset{1_U\otimes\iota}{\to} U \otimes R^m \overset{1_U\otimes\pi}{\to} U \otimes R^m/L \to 0$, we have $\sum_{j=1}^m(\sum_{k=1}^s a_kr_{kj}) \otimes e_j \in \mathrm{Ker}(1_U \otimes \pi) = \mathrm{Im}(1_U \otimes \iota)$, so there exists $u_1, \cdots, u_n \in U$ such that $\sum_{j=1}^m(\sum_{k=1}^s a_kr_{kj}) \otimes e_j = \sum_{i=1}^n u_i \otimes \alpha_i = \sum_{i=1}^n u_i \otimes (\sum_{j=1}^m c_{ij}e_j) = \sum_{j=1}^m(\sum_{i=1}^n u_ic_{ij}) \otimes e_j$, and so $\sum_{k=1}^s a_kr_{kj} = \sum_{i=1}^n u_ic_{ij}$. By (3), there exist $u_1', u_2', \cdots, u_n' \in U'$ such that $\sum_{k=1}^s a_kr_{kj} = \sum_{i=1}^n u_i'c_{ij}$, $j = 1, \cdots, m$. Thus $\sum_{k=1}^s a_k \otimes b_k = \sum_{i=1}^n u_i' \otimes (\sum_{j=1}^m c_{ij})\overline{e_j} = 0$ in $U' \otimes V$.

(5)⇔(6). It follows from the exact sequence

$$\mathrm{Hom}_R(A, U) \to \mathrm{Hom}_R(A, U/U') \to \mathrm{Ext}_R^1(A, U') \to \mathrm{Ext}_R^1(A, U). \qquad \square$$

Corollary 2.5. *Let $U_R' \leq U_R$. Then U' is I-$(1, \infty)$-pure in U if and only if $UT \cap U' = U'T$ for all finitely generated left ideals $T \subseteq I$.* □

Proposition 2.6 *Let $U'_R \leq U_R$. Then*

(1) *If U is n-generated, then U' is I-(m,n)-pure in U if and only if U' is I-(m,∞)-pure in U.*

(2) *If each finitely generated left ideal in I is n-generated, then U' is I-$(1,n)$-pure in U if and only if U' is I-$(1,\infty)$-pure in U.*

(3) *If each finitely generated right ideal in I is m-generated, then U' is I-$(m,1)$-pure in U if and only if U' is I-$(\infty,1)$-pure in U.*

Proof. (2) can be proved by Theorem 2.4(4), and (3) can be proved by Theorem 2.4(5). Now we prove only the necessity of (1).

Let $u_1, u_2, \cdots, u_n$ be a generating set of U. For every positive integer k and each $C \in I^{k\times m}$, if $x \in (U')^m \cap U^k C$, then $x = (u_1, u_2, \cdots, u_n)AC$ for some $A \in R^{n\times k}$. Since U' is $I-(m,n)$-pure in U, by Theorem 2.4(3), $x = (u'_1, u'_2, \cdots, u'_n)AC$ for some $u'_1, u'_2, \cdots, u'_n \in U$. So $x \in (U')^k C$, and thus $(U')^m \cap U^k C = (U')^k C$. Therefore U' is (m,k)-pure in U. □

Corollary 2.7 *Let $U'_R \leq U_R$. Then the following statements are equivalent*:

(1) *U' is I-pure in U.*

(2) *For every I-finitely presented left R-module V, the canonical map $\mathrm{Tor}_1^R(U,V) \to \mathrm{Tor}_1^R(U/U',V)$ is surjective.*

(3) *For any positive integers m, n and any $C \in I^{n\times m}$, $(U')^m \cap U^n C = (U')^n C$.*

(4) *For any positive integers m, n and any n-generated submodule T of ${}_R I^m$, $(U')^m \cap UT = U'T$*

.

(5) *For every I-finitely presented right R-module A, the canonical map $\mathrm{Hom}_R(A,U) \to \mathrm{Hom}_R(A,U/U')$ is surjective.*

(6) *For every I-finitely presented right R-module A, the canonical map $\mathrm{Ext}^1(A,U') \to \mathrm{Ext}^1(A,U)$ is a monomorphism.* □

Proposition 2.8 *Suppose E, F and G are right R-modules such that $E \subseteq F \subseteq G$. Then:*

(1) *If E is I-(m,n)-pure in F and F is I-(m,n)-pure in G, then E is I-(m,n)-pure in G.*

(2) *If E is I-(m,n)-pure in G, then E is I-(m,n)-pure in F.*

(3) *If F is I-(m,n)-pure in G, then F/E is I-(m,n)-pure in G/E.*

(4) *If E is I-(m,n)-pure in G and F/E is I-(m,n)-pure in G/E, then F is I-(m,n)-pure in G.*

Proof. (1) and (2) follows from the definition of I-(m,n)-pure submodules or Theorem 2.4(3).

(3). Let A be an I-(n,m)-presented right R-module. Since F is I-(m,n)-pure in G, by Theorem 2.4(5), the canonical map $\mathrm{Hom}_R(A,G) \stackrel{\alpha}{\to} \mathrm{Hom}_R(A,G/F)$ is surjective. Considering the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(A,G) & \stackrel{\alpha}{\longrightarrow} & \mathrm{Hom}_R(A,G/F) \\
\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
\mathrm{Hom}_R(A,G/E) & \stackrel{\tau}{\longrightarrow} & \mathrm{Hom}_R(A,(G/E)/(F/E))
\end{array}$$

, where σ is an isomorphism and hence a epimorphism, we have that the canonical map τ is epic. By Theorem 2.4(5), F/E is I-(m,n)-pure in G/E.

(4). Let V be an I-(n,m)-presented left R-module. Since E is I-(m,n)-pure in G, E is also I-(m,n)-pure in F, and so we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E\otimes V & \longrightarrow & F\otimes V & \longrightarrow & F/E\otimes V & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & E\otimes V & \longrightarrow & G\otimes V & \longrightarrow & G/E\otimes V & \longrightarrow & 0
\end{array}$$

. Since F/E is I-(m,n)-pure in G/E, g is monic. By five Lemma [9, 7.18], f is also monic, and thus F is I-(m,n)-pure in G. □

Corollary 2.9 *Suppose $E,\ F$ and G are right R-modules such that $E \subseteq F \subseteq G$.* Then:

(1) *If E is I-pure in F and F is I-pure in G, then E is I-pure in G.*

(2) *If E is I-pure in G, then E is I-pure in F.*

(3) *If F is I-pure in G, then F/E is I-pure in G/E.*

(4) *If E is I-pure in G and F/E is I-pure in G/E, then F is I-pure in G.* □

3 I-FP-injective Modules

Recall that a left R-module M is FP-injective if and only if every R-homomorphism from a finitely generated submodule of a free left R-module F to M extends to a homomorphism of F to M [1, Proposition 2.6] . FP-injective modules and their generalizations have been studied by many authors, for example, see [6, 7, 10, 11, 12, 13, 14]. Following [11], a left R-module M is called (m,n)-injective if every R-homomorphism from an n-generated submodule T of R^m to M extends to a homomorphism of R^m to M. It is easy to see that a left R-module M is FP-injective if and only if M is (m,n)-injective for each pair of positive integers m,n. Following [7], a left R-module M is called *F-injective* if every R-homomorphism from a finitely generated left ideal to M extends to a homomorphism of R to M. Following [10, 12], a left R-module M is called n*-injective* if every R-homomorphism from an n-generated left ideal to M extends to a homomorphism of R to M. Following [6], a left R-module M is called J*-injective* if every R-homomorphism from a finitely generated left ideal in $J(R)$ to M extends to a homomorphism of R to M. We extends the concepts of (m,n)-injective modules, FP-injective modules and J-injective modules as follows.

Definition 3.1. *A left R-module M is called I-(m,n)-injective, if every R-homomorphism from an n-generated submodule T of I^m to M extends to a homomorphism of R^m to M. A left R-module M is called I-FP-injective if M is I-(m,n)-injective for every pair of positive integers m, n. A left R-module M is called I-F-injective if M is I-(1,n)-injective for every positive integer n.*

It is easy to see that direct sums and direct summands of I-(m,n)-injective modules are I-(m,n)-injective. A left R-module M is (m,n)-injective if and only if M is R-(m,n)-injective, a left R-module M is FP-injective if and only if M is R-FP-injective, a left R-module M is J-injective if and only if M is J-F-injective. According to [15], a ring R is said to be left Soc-injective if every R-homomorphism from a semisimple submodule of ${}_RR$ to R extends to R. Clearly, if $Soc({}_RR)$ is finitely generated, then R is left Soc-injective if and only if ${}_RR$ is $Soc({}_RR)$-F-injective. Following [14], a left R-module M is called N-injective if $\mathrm{Ext}^1(R/T,M)=0$ for every finitely generated left ideal T in $Nil_*(R)$, where $Nil_*(R)$ is the prime radical of R, it is equal to the intersection of all the prime ideals in R [16]. It is clear that a left R-module M is N-injective if and only if M is $N(R)$-F-injective.

Theorem 3.2. *Let M be a left R-module. Then the following statements are equivalent:*

(1) *M is I-(m,n)-injective.*

(2) *$Ext^1(V,M)=0$ for every I-(m,n)-presented left R-module V.*

(3) $\mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1,...,\alpha_m\}=\alpha_1M+\cdots+\alpha_mM$ *for any m elements $\alpha_1,...,\alpha_m\in I_n$.*

(4) *If $x=(m_1,m_2,\ldots,m_n)'\in M_n$ and $A\in I^{n\times m}$ satisfy $\mathbf{l}_{R^n}(A)\subseteq\mathbf{l}_{R^n}(x)$, then $x=Ay$ for some $y\in M_m$.*

(5) $\mathbf{r}_{M_n}(R^nB\cap\mathbf{l}_{R^n}\{\alpha_1,...,\alpha_m\})=\mathbf{r}_{M_n}(B)+\alpha_1M+\cdots+\alpha_mM$ *for any m elements $\alpha_1,...,\alpha_m\in I_n$ and $B\in R^{n\times n}$.*

(6) *M is I-(m,1)-injective and $r_{M_m}(K\cap L)=r_{M_m}(K)+r_{M_m}(L)$, Where K and L are submodules of the left R-module I^m such that $K+L$ is n-generated.*

(7) *M is I-(m,1)-injective and $r_{M_m}(K \cap L) = r_{M_m}(K) + r_{M_m}(L)$, Where K and L are submodules of the left R-module I^m such that K is cyclic and L is $(n-1)$-generated.*

(8) *For each n-generated submodule T of I^m and any $f \in \mathrm{Hom}(T, M)$, if (α, g) is the pushout of (f, i) in the following diagram*

$$\begin{array}{ccc} T & \xrightarrow{\ i\ } & R^m \\ f\big\downarrow & & \big\downarrow g \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i is the inclusion map, there exists a homomorphism $h : P \to M$ such that $h\alpha = 1_M$.

(9) *M is absolutely I-(n,m)-pure, that is, M is I-(n,m)-pure in each module containing M.*

(10) *M is I-(n,m)-pure in E(M).*

(11) *M is an I-(n,m)-pure submodule of an I-(m, n)-injective module.*

Proof. $(1) \Leftrightarrow (2)$; $(8) \Rightarrow (1)$ and $(9) \Rightarrow (10), (11)$ are clear.

$(1) \Rightarrow (3)$. Always $\alpha_1 M + \cdots + \alpha_m M \subseteq \mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\}$. If $x \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\}$. Let A be the matrix with column vectors $\alpha_1, ..., \alpha_m$. Then the mapping $f : R^n A \to M; \beta A \mapsto \beta x$ is a well-defined left R-homomorphism. Since M is I-(m, n)-injective and $R^n A$ is an n-generated submodule of I^m, f can be extended to a homomorphism g of R^m to M. Now, for any $\beta \in R^n$, we have $\beta(\alpha_1 g(e_1) + \cdots + \alpha_m g(e_m)) = g(\beta A) = f(\beta A) = \beta x$, so $x = \alpha_1 g(e_1) + \cdots + \alpha_m g(e_m) \in \alpha_1 M + \cdots + \alpha_m M$. Thus $\mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\} \subseteq \alpha_1 M + \cdots + \alpha_m M$. Therefore, $\mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\} = \alpha_1 M + \cdots + \alpha_m M$.

$(3) \Rightarrow (1)$. Let $T = \sum_{i=1}^{n} R\beta_i$ be an n-generated submodule of I^m and f be a homomorphism from T to M. Write $u_i = f(\beta_i), i = 1, 2, \ldots, n, u = (u_1, u_2, \cdots, u_n)'$ and let A be the matrix with row vectors $\beta_1, ..., \beta_n$. Then $u \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}(A)$. By (3), there exists some $x_1, ..., x_m \in M$ such that $u = \alpha_1 x_1 + \cdots + \alpha_m x_m$, where $\alpha_1, ..., \alpha_m$ are column vectors of A. Now we define $g : R^m \to M; (r_1, \cdots, r_m) \mapsto r_1 x_1 + \cdots + r_m x_m$, then g is a left R-homomorphism, and it is easy to check that $f(\beta_i) = u_i = \beta_i(x_1, x_2, \cdots, x_m)' = g(\beta_i), i = 1, ..., n$, and so g extends f.

$(3) \Rightarrow (4)$. If $\mathbf{l}_{R^n}(A) \subseteq \mathbf{l}_{R^n}(x)$, where $A \in I^{n \times m}, x \in M_n$, then $x \in \mathbf{r}_{M_n}\mathbf{l}_{R^n}(x) \subseteq \mathbf{r}_{M_n}\mathbf{l}_{R^n}(A) = \alpha_1 M + \cdots + \alpha_m M$ by (3), where $\alpha_1, ..., \alpha_m$ are columns of A. Thus (4) is proved.

$(4) \Rightarrow (5)$. Let $x \in \mathbf{r}_{M_n}(R^n B \cap \mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\})$. Then $\mathbf{l}_{R^n}(BA) \subseteq \mathbf{l}_{R^n}(Bx)$, where A is the matrix whose column vectors are $\alpha_1, ..., \alpha_m$. By (4), $Bx = BAy$ for some $y \in M_m$. Hence $x - Ay \in \mathbf{r}_{M_n}(B)$, and so $x = z + Ay$ for some $z \in \mathbf{r}_{M_n}(B)$, proving that $\mathbf{r}_{M_n}(R^n B \bigcap \mathbf{l}_{R^n}(\alpha)) \subseteq \mathbf{r}_{M_n}(B) + \alpha_1 M + \cdots + \alpha_m M$. The other inclusion always holds.

$(5) \Rightarrow (3)$. By taking $B = E$ in (5).

$(1) \Rightarrow (6)$. Clearly, M is I-$(m, 1)$-injective and

$$r_{M_m}(K) + r_{M_m}(L) \subseteq r_{M_m}(K \cap L).$$

Conversely, let $x \in r_{M_m}(K \cap L)$. Then $f : K + L \to M$ is well defined by $f(k + l) = kx$ for all $k \in K$ and $l \in L$. Since M is I-(m, n)-injective, $f = \cdot y$ for some $y \in M_m$. Hence, for all $k \in K$ and $l \in L$, we have $ky = f(k) = kx$ and $ly = f(l) = 0$. Thus $x - y \in r_{M_m}(K)$ and $y \in r_{M_m}(L)$, so $x = (x - y) + y \in r_{M_m}(K) + r_{M_m}(L)$.

$(6) \Rightarrow (7)$ is trivial.

$(7) \Rightarrow (1)$. We proceed by induction on n. If $n = 1$, then (1) is clearly holds by hypothesis. Suppose $n > 1$. Let $T = R\beta_1 + R\beta_2 + \cdots + R\beta_n$ be an n-generated submodule of the left R-module I^m, $T_1 = R\beta_1$ and $T_2 = R\beta_2 + \cdots + R\beta_n$. Suppose $f : T \to M$ is a left R-homomorphism. Then $f|_{T_1} = \cdot y_1$ for some $y_1 \in M_m$ by hypothesis and $f|_{T_2} = \cdot y_2$ for some $y_2 \in M_m$ by induction hypothesis . Thus $y_1 - y_2 \in r_{M_m}(T_1 \cap T_2) = r_{M_m}(T_1) + r_M(T_2)$. So $y_1 - y_2 = z_1 + z_2$ for some $z_1 \in r_{M_m}(T_1)$ and $z_2 \in r_{M_m}(T_2)$. Let $y = y_1 - z_1 = y_2 + z_2$. Then for any $\beta \in T$, let $\beta = \beta_1 + \beta_2, \beta_1 \in T_1, \beta_2 \in T_2$, we have $\beta_1 z_1 = 0, \beta_2 z_2 = 0$. Hence $f(\beta) = f(\beta_1) + f(\beta_2) = \beta_1 y_1 + \beta_2 y_2 = \beta_1(y_1 - z_1) + \beta_2(y_2 + z_2) = \beta_1 y + \beta_2 y = \beta y$. So (1) follows.

$(1) \Rightarrow (8)$. Without loss of generality, we may assume that $P = (M \oplus R^m)/W$, where $W = \{f(a), -i(a)|a \in T\}, g(y) = (0, y) + W, \alpha(x) = (x, 0) + W$ for $x \in M$ and $y \in R^m$. Since M is I-(m, n)-injective, there is $\varphi \in \mathrm{Hom}_R(R^m, M)$ such that $\varphi i = f$. Define $h[(x, y) + W] = x + \varphi(y)$ for all $(x, y) + W \in P$. Then it is easy to check that h is well-defined and $h\alpha = 1_M$.

$(2) \Leftrightarrow (10)$. It follows from the exact sequence

$$\mathrm{Hom}_R(V, E(M)) \to \mathrm{Hom}_R(V, E(M)/M) \to \mathrm{Ext}^1_R(V, M) \to 0$$

and Theorem 2.4(5).

$(10) \Rightarrow (9)$. Suppose $M \leq M'$, then $M \leq E(M) \leq E(M')$. Since M is I-(n, m)-pure in $E(M)$ and $E(M)$ is pure in $E(M')$, M is I-(n, m)-pure in $E(M')$ by Proposition 2.8(1). Note that $M \leq M' \leq E(M')$, by Proposition 2.8(2), M is I-(n, m)-pure is M'.

$(11) \Rightarrow (10)$. Suppose that M is I-(n, m)-pure in M' and M' is I-(m, n)-injective. Then for every I-(n, m)-presented module ${}_RV$, since M is I-(n, m)-pure in M' and M' is I-(n, m)-pure in $E(M')$, $M \otimes V \to M' \otimes V$ and $M' \otimes V \to E(M') \otimes V$ are monomorphisms. Thus the following commutative diagram

$$\begin{array}{ccc} M \otimes V & \longrightarrow & M' \otimes V \\ \downarrow & & \downarrow \\ E(M) \otimes V & \longrightarrow & E(M') \otimes V \end{array}$$

gives that the map $M \otimes V \to E(M) \otimes V$ is a monomorphism, and so M is I-(n, m)-pure in $E(M)$. □

Corollary 3.3. *Let M be a left R-module. Then the following statements are equivalent:*

(1) *M is (m,n)-injective.*

(2) $\mathrm{Ext}^1(V, M) = 0$ *for every (m,n)-presented left R-module V.*

(3) $\mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\} = \alpha_1 M + \cdots + \alpha_m M$ *for any m elements* $\alpha_1, ..., \alpha_m \in R_n$.

(4) *If* $x = (m_1, m_2, \ldots, m_n)' \in M_n$ *and* $A \in R^{n\times m}$ *satisfy* $\mathbf{l}_{R^n}(A) \subseteq \mathbf{l}_{R^n}(x)$*, then* $x = Ay$ *for some* $y \in M_m$.

(5) $\mathbf{r}_{M_n}(R^n B \cap \mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\}) = \mathbf{r}_{M_n}(B) + \alpha_1 M + \cdots + \alpha_m M$ *for any m elements* $\alpha_1, ..., \alpha_m \in R_n$ *and* $B \in R^{n\times n}$.

(6) M *is (m,1)-injective and* $r_{M_m}(K \cap L) = r_{M_m}(K) + r_{M_m}(L)$*, Where* K *and* L *are submodules of the left R-modules* R^m *such that* $K + L$ *is n-generated.*

(7) M *is (m,1)-injective and* $r_{M_m}(K \cap L) = r_{M_m}(K) + r_{M_m}(L)$*, Where* K *and* L *are submodules of the left R-modules* R^m *such that* K *is cyclic and* L *is* $(n-1)$*-generated.*

(8) *For each n-generated submodule T of* R^m *and any* $f \in Hom(T, M)$*, if* (α, g) *is the pushout of* (f, i) *in the following diagram*

$$\begin{array}{ccc} T & \xrightarrow{\ i\ } & R^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i is the inclusion map, there exists a homomorphism $h : P \to M$ *such that* $h\alpha = 1_M$.

(9) *M is absolutely (n,m)-pure, that is, M is (n,m)-pure in each module containing M.*

(10) *M is (n,m)-pure in E(M).*

(11) *M is an (n,m)-pure submodule of an (m, n)-injective module.* □

We note that the equivalence of (1), (3), (6), (7) in Corollary 3.3 appears in [11, Corollary 2.5 and Corollary 2.10].

Corollary 3.4. *Let M be a left R-module. Then the following statements are equivalent:*

(1) *M is I-FP-injective.*

(2) $\mathrm{Ext}^1(V, M) = 0$ *for every I-finitely presented left R-module V.*

(3) *Every R-homomorphism from a finitely generated submodule of $I^{(\mathbb{N})}$ to M extends to a homomorphism of $R^{(\mathbb{N})}$ to M, where $\mathbb{N}$ is the set of all positive integers.*

(4) *For any positive integers m, n, $\mathbf{r}_{M_n}\mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\} = \alpha_1 M + \cdots + \alpha_m M$ for any m elements $\alpha_1, ..., \alpha_m \in I_n$.*

(5) *For any positive integers m, n, if $x = (m_1, m_2, \cdots, m_n)' \in M_n$ and $A \in I^{n\times m}$ satisfy $\mathbf{l}_{R^n}(A) \subseteq \mathbf{l}_{R^n}(x)$, then $x = Ay$ for some $y \in M_m$.*

(6) *For any positive integers m, n, $\mathbf{r}_{M_n}(R^n B \cap \mathbf{l}_{R^n}\{\alpha_1, ..., \alpha_m\}) = \mathbf{r}_{M_n}(B) + \alpha_1 M + \cdots + \alpha_m M$ for any m elements $\alpha_1, ..., \alpha_m \in I_n$ and $B \in R^{n\times n}$.*

(7) *For any positive integer m, M is I-(m,1)-injective and $r_{M_m}(K \cap L) = r_{M_m}(K) + r_{M_m}(L)$, Where K and L are submodules of the left R-module I^m such that $K + L$ is finitely generated.*

(8) *For any positive integer m, M is I-(m,1)-injective and $r_{M_m}(K \cap L) = r_{M_m}(K) + r_{M_m}(L)$, Where K and L are submodules of the left R-modules I^m such that K is cyclic and L is finitely generated.*

(9) *For each finitely generated submodule T of $I^{(\mathbb{N})}$ and any $f \in \mathrm{Hom}(T, M)$, if (α, g) is the pushout of (f, i) in the following diagram*

$$\begin{array}{ccc} T & \xrightarrow{\ i\ } & R^{(\mathbb{N})} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\ \alpha\ } & P \end{array}$$

where i is the inclusion map, there exists a homomorphism $h : P \to M$ such that $h\alpha = 1_M$.

(10) *M is absolutely I-pure, that is, M is I-pure in each module containing M.*

(11) *M is I-pure in E(M).*

(12) *M is an I-pure submodule of an I-FP-injective module.*

Proof. Since M is I-FP-injective if and only if M is I-(m, n)-injective for every pair of positive integers m, n, the equivalence of (1), (2),(4), (5), (6), (7), (8), (10), (11), (12) follows from Theorem 3.2.

$(1) \Leftrightarrow (3)$, and $(9) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (9)$ is similar to the proof of $(8) \Rightarrow (1)$ in Theorem 3.2. □

Proposition 3.5. *Let $\{M_\alpha\}_{\alpha\in A}$ be a family of left R-modules. Then the following statements are equivalent:*

(1). *Each M_α is I-(m, n)-injective.*

(2) $\prod_{\alpha\in A} M_\alpha$ *is I-(m, n)-injective .*

(3) $\oplus_{\alpha\in A} M_\alpha$ *is I-(m, n)-injective .*

Proof. It is trivial. □

Corollary 3.6. *Let $\{M_\alpha\}_{\alpha\in A}$ be a family of left R-modules. Then the following statements are equivalent:*

(1). *Each M_α is I-FP-injective.*

(2) $\prod_{\alpha\in A} M_\alpha$ *is I-FP-injective .*

(3) $\oplus_{\alpha\in A} M_\alpha$ *is I-FP-injective .* □

Recall that a submodule K of an R-module M is called small in M [9, 19.1], written $K << M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$. A ring R is called *semiregular* [17] if for any $a \in R$, R/Ra has a projective cover. A left R-module M is called *semiregular* [17] if for any $m \in M$, we have $M = P \oplus K$, where P is projective, $P \subseteq Rm$, and $Rm \cap K << K$. By [17, Lemma B.40, Lemma B.48], a ring R is semiregular if and only if the left R-module ${}_RR$ is semiregular.

Proposition 3.7. *If R is a semiregular ring, then a left R-module M is FP-injective if and only if it is J-FP-injective.*

Proof. Necessity is clear. To prove sufficiency, let N be a finitely generated submodule of a finitely generated free left R-module F and $f : N \to M$ be a left R-homomorphism. Since R is semiregular, by [17, Lemma B.54], F is semiregular. So, by [17, Lemma B.51], $F = P \oplus K$, where P is projective, $P \subseteq N$ and $N \cap K$ is small in K. Hence $F = N + K$, $N = P \oplus (N \cap K)$, and so $N \cap K$ is finitely generated. Since M is J-FP–injective, there exists a homomorphism $g : F \to M$ such that $g(x) = f(x)$ for all $x \in N \cap K$. Now let $h : F \to M; x \mapsto f(n) + g(k)$, where $x = n + k, n \in N, k \in K$. Then h is a well-defined left R-homomorphism and h extends f. □

4 I-flat Modules

Recall that a right R-module B is said to be *flat* if the functor $B \otimes_R$ is exact, it is well-known that a right R-module B is flat if and only if the canonical map $B \otimes T \to B \otimes R$ is monic for every finitely generated left ideal T, if and only if $\mathrm{Tor}_1(B, V) = 0$ for every finitely presented left R-module V. A right R-module B is said to be *n-flat* [10, 18], if for every n-generated left ideal T, the canonical map $V \otimes T \to V \otimes R$ is monic. 1-flat modules are also called P-flat by some authors [19, 20]. Following Zhang and Chen, a right R-module B is said to be *(m, n)-flat* [8] , if for every n-generated submodule T of the left R-module R^m, the canonical map $B \otimes T \to B \otimes R^m$ is monic. It is easy to see that a right R-module B is n-flat if and only if and only if it is $(1, n)$-flat, a right R-module B is flat if and only if and only if it is (m, n)-flat for each pair of positive integers m, n if and only if it is $(1, n)$-flat for each positive integer n. We extend the concepts of (m, n)-flat modules and flat modules respectively as follows.

Definition 4.1. *A right R-module B is said to be I-(m,n)-flat, if for every n-generated submodule T in I^m, the canonical map $B \otimes T \to B \otimes R^m$ is monic. A right R-module B is said to be I-flat in case it is I-(m,n)-flat for any positive integers m and n.*

Theorem 4.2. *For a right R-module B, the following statements are equivalent*:

(1) B *is I-(m,n)-flat.*

(2) $\mathrm{Tor}_1(B, R^m/T) = 0$ *for every n-generated submodule T of the left R-module I^m.*

(3) B^+ *is I-(m,n)-injective .*

(4) *For every n-generated submodule T of the left R-module I^m, the map $\mu_T : B \otimes T \to BT; \sum b_i \otimes a_i \mapsto \sum b_i a_i$ is a monomorphism.*

(5) *For all $X \in B^n, A \in I^{n \times m}$, if $XA = 0$, then exist positive integer l and $Y \in B^l, C \in R^{l \times n}$, such that $CA = 0$ and $X = YC$.*

Proof. $(1) \Leftrightarrow (2)$ follows from the exact sequence $0 \to \mathrm{Tor}_1(B, R^m/T) \to B \otimes T \to B \otimes R^m$.

$(2) \Leftrightarrow (3)$ follows from the isomorphism $\mathrm{Tor}_1(B, R^m/T)^+ \cong \mathrm{Ext}^1(R^m/T, B^+)$.

$(1) \Leftrightarrow (4)$. Consider the following commutative diagram

$$\begin{array}{ccc} B\otimes T & \xrightarrow{1_B\otimes i_T} & B\otimes R^m \\ \mu_T\downarrow & & \downarrow\sigma \\ BT & \xrightarrow{i_{VT}} & V^m \end{array}$$

, where $\sigma : b\otimes(r_1,\cdots,r_m)\mapsto(br_1,\cdots,br_m)$ is an isomorphism, and i_{VT} is the inclusion map. Then it is easy to see that $1_B\otimes i_T$ is monic if and only if μ_T is monic.

$(4)\Rightarrow(5)$. Let $X=(b_1,b_2,\cdots,b_n)$ and let $A_1,A_2,\cdots,A_n$ be the row vectors of A, $T=\sum_{j=1}^n RA_j$. Write e_j be the element in R^n with 1 in the jth position and 0's in all other positions, $j=1,2,\ldots,n$. Consider the short exact sequence

$$0\to K\overset{i_K}{\to}R^n\overset{f}{\to}T\to 0$$

where $f(e_j)=A_j$ for each $j=1,2,\ldots,n$. Since $XA=0$, by (4), $\sum_{j=1}^n(b_j\otimes f(e_j))=\sum_{j=1}^n(b_j\otimes A_j)=0$ as an element in $B\otimes_R T$. So in the exact sequence

$$B\otimes K\overset{1_B\otimes i_K}{\to}B\otimes R^n\overset{1_B\otimes f}{\to}B\otimes T\to 0$$

we have $\sum_{j=1}^n(b_j\otimes e_j)\in\mathrm{Ker}(1_B\otimes f)=\mathrm{Im}(1_B\otimes i_K)$. Thus there exist $u_h\in B,k_h\in K,h=1,2,\ldots,l$ such that $\sum_{j=1}^n(b_j\otimes e_j)=\sum_{h=1}^l(u_h\otimes k_h)$. Let $k_h=\sum_{j=1}^n c_{hj}e_j,h=1,2,\ldots,l$. Then $\sum_{j=1}^n c_{hj}a_j=\sum_{j=1}^n c_{hj}f(e_j)=f(k_h)=0,h=1,2,\ldots,l$. Write $C=(c_{hj})_{ln}$, then $CA=0$. Moreover, since $\sum_{j=1}^n(b_j\otimes e_j)=\sum_{h=1}^l(u_h\otimes k_h)=\sum_{h=1}^l(u_h\otimes(\sum_{j=1}^n c_{hj}e_j))=\sum_{j=1}^n((\sum_{h=1}^l u_hc_{hj})\otimes e_j)$, we have $b_j=\sum_{h=1}^l u_hc_{hj},j=1,2,\ldots,n$. Now, let $Y=(u_1,u_2,\cdots,u_l)$. Then $Y\in B^l$ and $X=YC$.

$(5)\Rightarrow(4)$. Let $T=\sum_{j=1}^n RX_j$ be an n-generated submodule of ${}_RI^m$ and suppose $A_i=\sum_{j=1}^n r_{ij}X_j\in T,b_i\in B$ with $\sum_{i=1}^k b_iA_i=0$. Then $\sum_{j=1}^n(\sum_{i=1}^k b_ir_{ij})X_j=0$. By (5), there exists elements $u_1,\ldots,u_m\in B$ and elements $c_{ij}\in R(i=1,\ldots,m,j=1,\ldots,n)$ such that $\sum_{j=1}^n c_{ij}X_j=0(i=1,\ldots,m)$ and $\sum_{i=1}^m u_ic_{ij}=\sum_{i=1}^k b_ir_{ij}(j=1,\ldots,n)$. Thus, $\sum_{i=1}^k b_i\otimes A_i=\sum_{i=1}^k b_i\otimes(\sum_{j=1}^n r_{ij}X_j)=\sum_{j=1}^n(\sum_{i=1}^k b_ir_{ij})\otimes X_j=\sum_{j=1}^n(\sum_{i=1}^m u_ic_{ij})\otimes X_j=\sum_{i=1}^m(u_i\otimes\sum_{j=1}^n c_{ij}X_j)=0$. And so (4) is proved. □

Corollary 4.3. *For a right R-module B, the following statements are equivalent*:

(1) B *is I-flat.*

(2) $\mathrm{Tor}_1(B,V)=0$ *for every I-finitely presented left R-module V.*

(3) B^+ *is I-FP-injective .*

(4) *For every positive integer m and every finitely generated submodule T of the left R-module I^m, the map $\mu_T : B\otimes T\to BT;\sum b_i\otimes a_i\mapsto\sum b_ia_i$ is a monomorphism.*

(5) *For any positive integers m, n and all $X\in B^n,A\in I^{n\times m}$, if $XA=0$, then exist positive integer l and $Y\in B^l,C\in R^{l\times n}$, such that $CA=0$ and $X=YC$.*

Remark 4.4. From Corollary 4.3, the I-flatness of B_R can be characterized by the I-FP-injectivity of B^+. On the other hand, by [5, Lemma 2.7(1)], the sequence $\mathrm{Tor}_1(B^+,V)\to\mathrm{Ext}^1(V,B)^+\to 0$ is exact for all finitely presented left R-module V, so if B^+ is I-flat, then B is I-FP-injective.

Proposition 4.5. *If R is a semiregular ring, then a right R-module B is flat if and only if it is J-flat.*

Proof. Clearly, flat module is J-flat. Conversely, if B is J-flat, then by Corollary 4.3, B^+ is J-FP-injective. But R is a semiregular ring, by Proposition 3.7, B^+ is FP-injective, and so B is flat.

Proposition 4.6. *Let $U'_R \leq U_R$.*
(1) *If U/U' is I-(m,n)-flat, then U' is I-(m,n)-pure in U.*
(2) *If U' is I-(m,n)-pure in U and U is I-(m,n)-flat, then U/U' is I-(m,n)-flat.*

Proof. It follows from the exact sequence

$$\mathrm{Tor}_1(U, R^m/T) \to \mathrm{Tor}_1(U/U', R^m/T) \to U' \otimes R^m/T \to U \otimes R^m/T$$

and Theorem 4.2(2). □

Corollary 4.7. *Let F be an I-(m,n)-flat module and K a submodule of F. Then F/K is I-(m,n)-flat if and only if K is I-(m,n)-pure in F.* □

The results of following Corollary 4.8 are well-known.

Corollary 4.8. *Let F be a flat module and K a submodule of F. Then the following statements are equivalent:*
(1) *F/K is flat.*
(2) *$K \cap FT = KT$ for every finitely generated left ideal T.*
(3) *$K \cap FT = KT$ for every left ideal T.*

Proof. $(1) \Leftrightarrow (2)$. Since a module is flat if and only if it is R-$(1, \infty)$ flat, so , by Corollary 4.7. F/K is flat if and only if K is R-$(1, \infty)$-pure in F. Thus, by Theorem 2.4(4), we have that F/K is flat if and only if $K \cap FT = KT$ for every finitely generated left ideal T.
$(2) \Leftrightarrow (3)$. It is obvious. □

Corollary 4.9. *I-(n,m)-presented I-(m,n)-flat module is projective.*

Proof. By Proposition 4.6(1) and Theorem 2.4(5). □

Corollary 4.10. *I-finitely presented I-flat module is projective. In particular, finitely presented flat module is projective, and J-finitely presented J-flat module is projective.* □

Theorem 4.11. *Every pure submodule of an I-(m,n)-flat module is I-(m,n)-flat. In particular, every pure submodule of an (m,n)-flat module is (m,n)-flat.*

Proof. Let A be a pure submodule of an I-(m, n)-flat right R-module B. Then the pure exact sequence $0 \to A \to B \to B/A \to 0$ induces a split exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Since B is I-(m, n)-flat, by Theorem 4.2, B^+ is I-(m, n)-injective, and so A^+ is I-(m, n)-injective. Thus A is I-(m, n)-flat by Theorem 4.2 again. □

Corollary 4.12. *Every pure submodule of an I-flat module is I-flat.* □

Proposition 4.13. *Let $\{M_\alpha\}_{\alpha \in A}$ be a family of right R-modules. Then $\oplus_{\alpha \in A} M_\alpha$ is I-flat if and only if each M_α is I-flat.*

Proof. It follows from the isomorphism $\mathrm{Tor}_1(\oplus_{\alpha \in A} M_\alpha, N) \cong \oplus_{\alpha \in A} \mathrm{Tor}_1(M_\alpha, N)$. □

5 I-coherent Rings and I-semihereditary Rings

Recall that a ring R is called left coherent if every finitely generated left ideal of R is finitely presented, a ring R is called left J-coherent [6] if every finitely generated left ideal in J is finitely presented, a ring R is called left Nil_*-coherent [14] if every finitely generated left ideal in $Nil_*(R)$ is finitely presented. We extend these concepts as follows.

Definition 5.1. *Let R be a ring and I be an ideal of R. Then R is called left I-coherent if every finitely generated left ideal in I is finitely presented.*

Following [21], a ring R is called *left min-coherent* if every minimal left ideal of R is finitely presented.

Example 5.2. *A ring R is left min-coherent if and only if R is left $Soc(_RR)$-coherent.*

We note that since left J-coherent rings need not be left coherent [6, Example 2.8], and left min-coherent rings need not be left coherent [21, Remark 4.2(1)]. So, a left I-coherent ring need not be left coherent for any ideal I.

Recall that a left R-module A is called 2-presented if there exists an exact sequence $F_2 \to F_1 \to F_0 \to A \to 0$ in which every F_i is a finitely generated free module.

Theorem 5.3. *Let R be a ring and I be an ideal of R. Then the following statements are equivalent:*

(1) *R is a left I-coherent ring.*

(2) *For every positive integer m, every finitely generated submodule A of the left R-module I^m is finitely presented.*

(3) *Every I-finitely presented left R-module is 2-presented.*

Proof. $(1) \Rightarrow (2)$. We prove by induction on m. If $m = 1$, then A is a finitely generated left ideal in I, by hypothesis, A is finitely presented. Assume that every finitely generated submodule of the left R-module I^{m-1} is finitely presented. Then for any finitely generated submodule A of the left R-module I^m. Let $B = A \cap (Re_1 \oplus \cdots \oplus Re_{m-1})$. Then each $a \in A$ has a unique expression $a = b + re_m$, where $b \in Re_1 \oplus \cdots \oplus Re_{m-1}, r \in R$, where $e_j \in R^m$ with 1 in the jth position and 0's in all other positions. If $\varphi : A \to R$ is defined by $a \mapsto r$, then there is an exact sequence $0 \to B \to A \overset{\varphi}{\to} L \to 0$, where $L = \mathrm{Im}(\varphi)$ is a finitely generated left ideal in I. By hypothesis, L is finitely presented, and so B is finitely generated. Since B is contained in I^{m-1}, the induction hypothesis gives B is finitely presented. Therefore, A is also finitely presented by [9, 25.1(2)(ii)].

$(2) \Rightarrow (1)$, and $(2) \Leftrightarrow (3)$ are obvious. □

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [22], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-precover and an $\mathcal{F}$-cover. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 5.4. *Let R be a ring and I be an ideal of R. Then the following statements are equivalent:*

(1) *R is left I-coherent.*

(2) $\varinjlim \mathrm{Ext}^1_R(V, M_\alpha) \cong \mathrm{Ext}^1_R(V, \varinjlim M_\alpha)$ *for any I-finitely presented left R-module* V *and direct system* $(M_\alpha)_{\alpha\in A}$ *of left R-modules.*

(3) $\mathrm{Tor}^R_1(\prod N_\alpha, V) \cong \prod \mathrm{Tor}^R_1(N_\alpha, V)$ *for any family* $\{N_\alpha\}$ *of right R-modules and any I-finitely presented left R-module* V*.*

(4) *Any direct product of copies of* R_R *is* I*-flat.*

(5) *Any direct product of* I*-flat right R-modules is* I*-flat.*

(6) *Any direct limit of* I*-*FP*-injective left R-modules is* I*-*FP*-injective.*

(7) *Any direct limit of injective left R-modules is* I*-*FP*-injective.*

(8) *A left R-module M is* I*-*FP*-injective if and only if* M^+ *is* I*-flat.*

(9) *A left R-module M is* I*-*FP*-injective if and only if* M^{++} *is* I*-*FP*-injective.*

(10) *A right R-module M is* I*-flat if and only if* M^{++} *is* I*-flat.*

(11) *For any ring S,* $\mathrm{Tor}^R_1(\mathrm{Hom}_S(B, E), V) \cong \mathrm{Hom}_S(\mathrm{Ext}^1_R(V, B), E)$ *for the situation* $({}_RV, {}_RB_S, E_S)$ *with* V *I-finitely presented and* E_S *injective.*

(12) *Every right R-module has an* I*-flat preenvelope.*

Proof. $(1) \Rightarrow (2)$ follows from [5, Lemma 2.9(2)].

$(1) \Rightarrow (3)$ follows from [5, Lemma 2.10(2)].

$(2) \Rightarrow (6) \Rightarrow (7)$, $(3) \Rightarrow (5) \Rightarrow (4)$ are trivial.

$(7) \Rightarrow (1)$. Let $V = R^m/T$ be an I-finitely presented left R-module, where T be a finitely generated submodule of I^m, and let $(M_\alpha)_{\alpha\in A}$ a direct system of FP-injective left R-modules (with A directed). Then $\varinjlim M_\alpha$ is I-FP-injective by (7), and so $\mathrm{Ext}^1(V, \varinjlim M_\alpha) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Hom}(V, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(R^m, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Hom}(T, M_\alpha) & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
\mathrm{Hom}(V, \varinjlim M_\alpha) & \longrightarrow & \mathrm{Hom}(R^m, \varinjlim M_\alpha) & \longrightarrow & \mathrm{Hom}(T, \varinjlim M_\alpha) & \longrightarrow & 0.
\end{array}$$

Since f and g are isomorphism by [9, 25.4(d)], h is also an isomorphism by the Five Lemma. So T is finitely presented by [9, 25.4(e)] and then V is 2-presented. Hence R is left I-coherent.

$(4) \Rightarrow (1)$. Let T be a finitely generated submodule of the left R-module I^m. By (4), $\mathrm{Tor}_1(\Pi R, R^m/T) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Pi R) \otimes T & \longrightarrow & (\Pi R) \otimes R^m & \longrightarrow & (\Pi R) \otimes R^m/T & \longrightarrow & 0 \\
& & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
0 & \longrightarrow & \Pi T & \longrightarrow & \Pi R^m & \longrightarrow & \Pi(R^m/T) & \longrightarrow & 0
\end{array}$$

Since f_2 and f_3 are isomorphism by [22, Theorem 3.2.22], f_1 is an isomorphism by the Five Lemma. So T is finitely presented by [22, Theorem 3.2.22] again. Hence R is left I-coherent.

$(5) \Rightarrow (12)$. Let N be any right R-module. By [22, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ dependent on Card(N) and Card(R) such that for any homomorphism $f : N \to F$ with F I-flat, there is a pure submodule S of F such that $f(N) \subseteq S$ and Card $S \le \aleph_\alpha$. Thus f has a factorization $N \to S \to F$ with S I-flat by Corollary 4.12. Now let $\{\varphi_\beta\}_{\beta\in B}$ be all such homomorphisms $\varphi_\beta : N \to S_\beta$ with Card $S_\beta \le \aleph_\alpha$ and S_β I-flat. Then any homomorphism $N \to F$ with F I-flat has a factorization $N \to S_i \to F$ for some $i \in B$. Thus the homomorphism $N \to \Pi_{\beta\in B} S_\beta$ induced by all φ_β is an I-flat preenvelope since $\Pi_{\beta\in B} S_\beta$ is I-flat by (5).

$(12) \Rightarrow (5)$ follows from [23, Lemma 1].

$(1) \Rightarrow (11)$. Let V be any I-finitely presented left R-module. Since R is left I-coherent, V is 2-presented. And so (11) follows from [5, Lemma 2.7(2)].

$(11) \Rightarrow (8)$. Let $S = \mathbb{Z}, C = \mathbb{Q}/\mathbb{Z}$ and $B = M$. Then $\mathrm{Tor}_1(M^+, V) \cong \mathrm{Ext}^1(V, M)^+$ for any I-finitely presented left R-module V by (11), and hence (8) holds.

$(8) \Rightarrow (9)$. Let M be a left R-module. If M is I-FP-injective, then M^+ is I-flat by (8), and so M^{++} is I-FP-injective by Corollary 4.3. Conversely, if M^{++} is I-FP-injective, then M, being a pure submodule of M^{++} (see [24, Exercise 41, p.48]), is I-FP-injective by Corollary 3.4.

$(9) \Rightarrow (10)$. If M is an I-flat right R-module, then M^+ is an I-FP-injective left R-module by Corollary 4.3, and so M^{+++} is I-FP-injective by (9). Thus M^{++} is I-flat by Corollary 4.3 again. Conversely, if M^{++} is I-flat, then M is I-flat by Corollary 4.12 since M is a pure submodule of M^{++}.

$(10) \Rightarrow (5)$. Let $\{N_\alpha\}_{\alpha\in A}$ be a family of I-flat right R-modules. Then by Proposition 4.13, $\oplus_{\alpha\in A}N_\alpha$ is I-flat, and so $(\Pi_{\alpha\in A}N_\alpha^+)^+ \cong (\oplus_{\alpha\in A}N_\alpha)^{++}$ is I-flat by (10). Since $\oplus_{\alpha\in A}N_\alpha^+$ is a pure submodule of $\Pi_{\alpha\in A}N_\alpha^+$ by [25, Lemma 1(1)], $(\Pi_{\alpha\in A}N_\alpha^+)^+ \to (\oplus_{\alpha\in A}N_\alpha^+)^+ \to 0$ splits, and hence $(\oplus_{\alpha\in A}N_\alpha^+)^+$ is I-flat. Thus $\Pi_{\alpha\in A}N_\alpha^{++} \cong (\oplus_{\alpha\in A}N_\alpha^+)^+$ is I-flat. Since $\Pi_{\alpha\in A}N_\alpha$ is a pure submodule of $\Pi_{\alpha\in A}N_\alpha^{++}$ by [25, Lemma 1(2)], $\Pi_{\alpha\in A}N_\alpha$ is I-flat by Corollary 4.12. □

Corollary 5.5. *Let R be a left I-coherent ring. Then every left R-module has an I-FP-injective cover.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B I-FP-injective. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is split. Since R is left I-coherent, B^+ is I-flat by Theorem 5.4, so C^+ is I-flat, and hence C is I-FP-injective by Remark 4.4. Thus, the class of I-FP-injective modules is closed under pure quotients. By [26, Theorem 2.5], every left R-module has an I-FP-injective cover. □

Recall that a ring R is called left semihereditary if every finitely generated left ideal of R is projective, a ring R is called left J-semihereditary [6] if every finitely generated left ideal in J is projective. We extend these concepts as follows.

Definition 5.6. *Let R be a ring and I be an ideal of R. Then R is called left I-semihereditary if every finitely generated left ideal in I is projective.*

Example 5.7. *Recall that a ring R is called left PS [27] if every minimal left ideal of R is projective. It is easy to see that a ring R is left PS if and only if R is left $Soc(_RR)$-semihereditary.*

Let R be a non-coherent commutative domain and G a free abelian group with $rank\ G = \infty$. Then the group ring RG is left J-semihereditary but not left semihereditary (see [6, p.152]). So, a let I-semihereditary ring need not be left semihereditary for a general ideal I.

Theorem 5.8. *Let R be a ring and I be an ideal of R. Then the following statements are equivalent:*

(1) *R is a left I-semihereditary ring.*

(2) *For every positive integer m, every finitely generated submodule A of the left R-module I^m is projective.*

(3) *If $0 \to K \to P \to V \to 0$ is exact , where V is I-finitely presented , P is finitely generated projective and K is finitely generated, then K is projective.*

Proof. $(1) \Rightarrow (2)$. We prove by induction on m. If $m = 1$, then A is a finitely generated left ideal in I, by hypothesis, A is projective. Assume that every finitely generated submodule of the left R-module I^{m-1} is projective. Then for any finitely generated submodule A of the left R-module I^m. Let $B = A \cap (Re_1 \oplus \cdots \oplus Re_{m-1})$. Then each $a \in A$ has a unique expression $a = b + re_m$, where $b \in Re_1 \oplus \cdots \oplus Re_{m-1}, r \in R$, where $e_j \in R^m$ with 1 in the jth position and 0's in all other positions. If $\varphi : A \to R$ is defined by $a \mapsto r$, then there is an exact sequence $0 \to B \to A \xrightarrow{\varphi} L \to 0$, where $L = \mathrm{Im}(\varphi)$ is a finitely generated left ideal in I. By hypothesis, L is projective, so $A \cong B \oplus L$ and then B is finitely generated. Since B is contained in I^{m-1}, the induction hypothesis gives B, hence A, is projective.

$(2) \Rightarrow (1)$. It is clear.

$(2) \Leftrightarrow (3)$. By the dual of Schanuel's lemma [9, 50.2(1)]. □

Corollary 5.9. *If R is a left J-semihereditary ring, then for every positive integer m, every finitely generated submodule of the left R-module J^m is projective.*

Corollary 5.10. *If R is a left semihereditary ring, then every finitely generated submodule of a projective left R-module is projective.*

Theorem 5.11. *The following statements are equivalent for a ring R:*

(1) *R is a left I-semihereditary ring.*
(2) *R is left I-coherent and every submodule of an I-flat right R-module is I-flat.*
(3) *R is left I-coherent and every right ideal is I-flat.*
(4) *R is left I-coherent and every finitely generated right ideal is I-flat.*
(5) *Every quotient module of an I-FP-injective left R-module is I-FP-injective.*
(6) *Every quotient module of an injective left R-module is I-FP-injective.*
(7) *Every left R-module has a monic I-FP-injective cover.*
(8) *Every right R-module has an epic I-flat envelope.*

Proof. (2)⇒(3)⇒ (4), and (5)⇒(6) are trivial.

(1)⇒(2). Let $V = R^m/L$ be an I-finitely presented left R-module, where L is a finitely generated submodule of I^m. Then by Theorem 5.8, L is projective, and so finitely presented, it shows that V is 2-presented, and thus R is left I-coherent. Let A be a submodule of an I-flat right R-module B, and let m be any positive and T a finitely generated submodule of ${}_RI^m$. Then T is projective by Theorem 5.8 again, and hence T is flat. So the exactness of $0 = \mathrm{Tor}_2(B/A, R^m) \to \mathrm{Tor}_2(B/A, R^m/T) \to \mathrm{Tor}_1(B/A, T) = 0$ implies that $\mathrm{Tor}_2(B/A, R^m/T) = 0$. And thus from the exactness of the sequence $0 = \mathrm{Tor}_2(B/A, R^m/T) \to \mathrm{Tor}_1(A, R^m/T) \to \mathrm{Tor}_1(B, R^m/T) = 0$ we have $\mathrm{Tor}_1(A, R^m/T) = 0$, it follows that A is I-flat.

(4)⇒(1). Let T be a finitely generated left ideal in I. Then for any finitely generated right ideal K of R, the exact sequence $0 \to K \to R \to R/K \to 0$ implies the exact sequence $0 \to \mathrm{Tor}_2(R/K, R/T) \to \mathrm{Tor}_1(K, R/T) = 0$ since K is I-flat. So $\mathrm{Tor}_2(R/K, R/T) = 0$, and hence we obtain an exact sequence $0 = \mathrm{Tor}_2(R/K, R/T) \to \mathrm{Tor}_1(R/K, T) \to 0$. Thus, $\mathrm{Tor}_1(R/K, T) = 0$. Note that T is finitely presented for R is left I-coherent, so T is a finitely presented flat left R-module. Therefore, T is projective.

(1)⇒(5). Let M be an I-FP-injective left R-module and N be a submodule of M. Then for any positive integer m and finitely generated submodule T of ${}_RI^m$, since T is projective, the exact sequence $0 = \mathrm{Ext}^1(T, N) \to \mathrm{Ext}^2(R^m/T, N) \to \mathrm{Ext}^2(R^m, N) = 0$ implies that $\mathrm{Ext}^2(R^m/T, N) = 0$. Thus the exact sequence $0 = \mathrm{Ext}^1(R^m/T, M) \to \mathrm{Ext}^1(R^m/T, M/N) \to \mathrm{Ext}^2(R^m/T, N) = 0$ implies that $\mathrm{Ext}^1(R^m/T, M/N) = 0$. Consequently, M/N is I-FP-injective.

(6)⇒(1). Let T be a finitely generated left ideal in I. Then for any left R-module M, by (6), $E(M)/M$ is I-FP-injective , and so $\mathrm{Ext}^1(R/T, E(M)/M) = 0$. Thus, the exactness of the sequence $0 = \mathrm{Ext}^1(R/T, E(M)/M) \to \mathrm{Ext}^2(R/T, M) \to \mathrm{Ext}^2(R/T, E(M)) = 0$ implies that $\mathrm{Ext}^2(R/T, M) = 0$. And so, the exactness of the sequence $0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(T, M) \to \mathrm{Ext}^2(R/T, M) = 0$ implies that $\mathrm{Ext}^1(T, M) = 0$, this follows that T is projective, as required.

(2), (5)⇒(7). Since R is left I-coherent by (2), for any left R-module M, there is an I-FP-injective cover $f : E \to M$ by Corollary 5.4. Note that $\mathrm{Im}(f)$ is I-n-injective by (5), and $f : E \to M$ is an I-FP-injective precover, so for the inclusion map $i : \mathrm{Im}(f) \to M$, there is a homomorphism $g : \mathrm{Im}(f) \to E$ such that $i = fg$. Hence $f = f(gf)$. Observing that $f : E \to M$ is an I-FP-injective cover and

gf is an endomorphism of E, so gf is an automorphisms of E, and thus $f : E \to M$ is a monic I-FP-injective cover.

(7)⇒(5). Let M be an I-FP-injective left R-module and N be a submodule of M. By (7), M/N has a monic I-FP-injective cover $f : E \to M/N$. Let $\pi : M \to M/N$ be the natural epimorphism. Then there exists a homomorphism $g : M \to E$ such that $\pi = fg$. Thus f is an isomorphism, and so $M/N \cong E$ is I-FP-injective.

(2)⇔(8). By Theorem 5.4 and [23, Theorem 2]. □

Corollary 5.12. *The following statements are equivalent for a ring R:*

(1) *R is a left semihereditary ring.*
(2) *R is left coherent and every submodule of a flat right R-module is flat.*
(3) *R is left coherent and every right ideal is flat.*
(4) *R is left coherent and every finitely generated right ideal is flat.*
(5) *Every quotient module of an FP-injective left R-module is FP-injective.*
(6) *Every quotient module of an injective left R-module is FP-injective.*
(7) *Every left R-module has a monic FP-injective cover.*
(8) *Every right R-module has an epic flat envelope.* □

Corollary 5.13. *The following statements are equivalent for a ring R:*

(1) *R is a left J-semihereditary ring.*
(2) *R is left J-coherent and every submodules of a J-flat right R-modules is flat.*
(3) *R is left J-coherent and every right ideal is J-flat.*
(4) *R is left J-coherent and every finitely generated right ideal is J-flat.*
(5) *Every quotient module of an J-FP-injective left R-module is J-FP-injective.*
(6) *Every quotient module of an injective left R-module is J-FP-injective.*
(7) *Every left R-module has a monic J-FP-injective cover.*
(8) *Every right R-module has an epic J-flat envelope.* □

6 Conclusion

Let R be a ring and I an ideal of R . In this paper, we define and study I-pure submodules , I-FP-injective modules, I-flat modules , I-coherent rings and I-semihereditary rings, a series of interesting results are obtained, some results generalize the well-known results on pure submodules , FP-injective modules, flat modules , coherent rings and semihereditary rings, respectively.

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Competing Interests

The author declares that no competing interests exist.

References

[1] Stenström B. Coherent rings and FP-injective modules. J. London Math. Soc. 1970;2(2):323-329.

[2] Megibben C. Absolutely pure modules. Proc. Amer. Math. Soc. 1970;26(4):561-566.

[3] Chase SU. Direct products of modules. Trans. Amer. Math. Soc. 1960;97(3):457-473.

[4] Cartan H, Eilenberg S. Homological algebra. Princeton University Press, Princeton; 1956.

[5] Chen JL, Ding NQ. On n-coherent rings. Comm. Algebra. 1996;24(10):3211-3216.

[6] Ding NQ, Li YL, Mao LX. J-coherent rings. J. Algebra Appl. 2009;8(2):139-155.

[7] Gupta RN. On f-injective modules and semihereditary rings. Proc. Nat. Inst. Sci. India Part A. 1969;35(2):323-328.

[8] Zhang XX, Chen JL. On (m, n)-injective modules and (m, n)-coherent rings. Algebra colloq. 2005;12(1):149-160.

[9] Wisbauer R. Foundations of Module and Ring Theory, Algebra, Logic and Applications, vol.3, Gordon and Breach Science, Pennsylvania; 1991.

[10] Ahmad S. n-injective and n-flat modules. Comm. Algebra. 2001;29(5):2039-2050.

[11] Chen JL, Ding NQ, Li YL, Zhou YQ. On (m, n)-injectivity of modules. Comm. Algebra. 2001;29(12):5589-5603.

[12] Nicholson WK, Yousif MF. Principally injective rings. J. Algebra. 1995;174(1):77-93.

[13] Yousif MF, Zhou YQ. Rings for which certain elements have the principal extension property. Algebra colloq. 2003;10(4):501-512.

[14] Xiang YM, Ouyang LQ. Nil_*-Coherent rings. Bull. Korean Math. Soc. 2014;51(2):579-594.

[15] Amin I, Yousif M, Zeyada N. Soc-injective rings and modules. Comm. Algebra. 2005;33(11):4229-4250.

[16] Lam TY. Lectures on Modules and Rings, Graduate Texts in Mathematic vol.131, Springer-Verlag, New York; 2001.

[17] Nicholson WK, Yousif MF. Quasi-Frobenius Rings, Cambridge University Press, Cambridge; 2003.

[18] Dobbs DD. On n-flat modules over a commutative ring. Bull. Austral. Math. Soc. 1991;43(3):491-498.

[19] Cheniour F, Mahdou N. When every P-flat ideal is flat. Bull. Iranian Math. Soc. 2014;37(3):251-267.

[20] Couchot F. Flat modules over valuation rings. J. Pure Appl. Algebra. 2007;211(1):235-247.

[21] Mao LX. Min-flat modules and Min-coherent rings. Comm. Algebra. 2007;35(2):635-650.

[22] Enochs EE, Jenda OMG. Relative Homological Algebra, Walter de Gruyter, Berlin-New York; 2000.

[23] Chen JL, Ding NQ. A note on existence of envelopes and covers. Bull. Austral. Math. Soc. 1996;54(3):383-390.

[24] Stenström B. Rings of Quotients, Springer-Verlag, Berlin-Heidelberg-New York; 1975.

[25] Cheatham TJ, Stone DR. Flat and projective character modules, Proc. Amer. Math. Soc. 1981;81(2):175-177.

[26] Holm H, Jϕrgensen P. Covers, precovers, and purity. Illinois J. Math. 2008;52(2):691-703.

[27] Nicholson WK, Watters JF. Rings with projective socle. Proc. Amer. Math. Soc. 1988;102(3):443-450.

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