The Modified Variational Iteration Method to Solve Linear Fractional Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

Method Article

Abstract

In this work we conceive a method of how the Lagrange multiplier of modified Variational Iteration Method can be defined from Laplace transform, And we use this technique to solve both differential equations and FDEs with initial value conditions, With Illustrative examples by applying the modified VIM to both Ordinary differential equations and fractional Differential Equations.

1 Introduction

Many Method have been used to solve a number of nonlinear problems which arise in mathematical physics and another related areas like [1-4]. The technique of Lagrange multipliers [5,6,7,8,2,3] was widely used to solve a number of nonlinear problems and another related areas, and it was developed into a powerful analytical method, Like the varianional iteration method [9-17], for solving differential equations. The method has been applied to initial boundary problems [6-8], fractal initial value problems [18-20], etc. Generally, in applications

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of variational iteration method to initial value problems of differential equations, one usually follows the fallowing three steps:

(a) Establishing the correction functional
(b) Identifying the Lagrange multipliers
(c) Determining the initial iteration

The step (b) is very decisive. Applications of the method to fractional differential equations (FDEs) mainly and directly used the Lagrange multipliers in ordinary differential equations (ODEs).the present article conceives a method how the Lagrange multiplier has to be defined from Laplace transform. This technique can be easily extended to solve both differential equations and FDEs with initial value conditions.

2 Basics of the Variation Iteration Method

To illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{d^m u(t)}{dt^m} + R[u(t)] + N[u(t)] = g(t) \tag{1}$$

Where R is a linear operator and N is a nonlinear operator and $g(t)$ is a given continuous function and $\frac{d^m u}{dt^m}$ is the term of the highest-order derivative.

The basic concept of the method is to construct a correction functional for the system (1), which reads

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda(t,\tau)\{Ru_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau, \tag{2}$$

Where $\lambda(t,\tau)$ is a general Lagrange multipliers [14,15,5] that can be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta u_n = 0$, where δ is the variational derivative.

To illustrate how restricted variation works in the variational iteration method.

3 New Identification of the Lagrange Multipliers

Now we revisit the original idea of the Lagrange multipliers in the case of an algebraic equation. Firstly, an iteration formula for finding the solution of the algebraic equation $f(x) = 0$ can be constructed as:

$$x_{n+1} = x_n + \lambda f(x_n) \tag{3}$$

The optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$ leads to

$$\lambda = -\frac{1}{f'(x_n)} \tag{4}$$

Where δ is the classical variational operator. From (3) and (4), for a given initial value x_0 we can find the approximate solution x_{n+1} by the iterative scheme for (4)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ f'(x_n) \neq 0, n = 0,1,2,\dots \tag{5}$$

The algorithm is well known as the Newton-Raphson method and has quadratic convergence.

Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the Laplace transform [6] to Equation (1), then the linear part is transformed into an algebraic equation as follows:

$$s^m u(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)} u(0) + L[R[u]] + L[N[u]] - L[g(t)] = 0 \tag{6}$$

where

$$u(s) = L[u(t)] = \int_0^\infty e^{-st} u(t) dt.$$

The iteration formula of (6) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$u_{n+1}(s) = u_n(s) + \lambda(s)[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)} u(0) + L(R[u_n] + N[u_n] - g(t))] \tag{7}$$

Considering $L(R[u_n] + N[u_n])$ as restricted terms, one can derive a Lagrange multiplier as:

$$\delta u_{n+1}(s) = \delta u_n(s) + s^m \delta\lambda(s) u_n(s)$$
$$0 = 1 + s^m \lambda(s)$$
$$\lambda = -\frac{1}{s^m} \tag{8}$$

With Equation (8) and the inverse-Laplace transform L^{-1},

The iteration formula (7) can be explicitly given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^m}\left[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)} u(0) + L(R[u_n] + N[u_n] - g(t))\right]\right]$$

$$= L^{-1}\left(\frac{1}{s^m} u^{(m-1)}(0) + \cdots + \frac{u(0)}{s} - \frac{1}{s^m} L(R[u_n] + N[u_n] - g(t))\right) \tag{9}$$

Where the initial iteration $u_0(t)$ can be determined by

$$u_0(t) = L^{-1}\left(\frac{1}{s^m} u^{(m-1)}(0) + \cdots + \frac{u(0)}{s}\right)$$
$$= u(0) + u'(0)t + \cdots + \frac{u^{(m-1)} t^{m-1}}{(m-1)!} \tag{10}$$

Equation (10) also explained why the initial iteration in the classical VIM is determined by the Taylor series. So Consequently, the solution

$$u(t) = \lim_{n\to\infty} u_n(t).$$

4 The Caputo Differintegral

Let a, T, α be real constants ($\alpha < T$), $n_c = \max(0, -[-\alpha])$ and $f(t)$ a function which is integrable on $\langle a, T \rangle$ in case $n_c = 0$ and n_c-times differentiable on $\langle a, T \rangle$ except on a set of measure zero in case $n_c > 0$. Then the Caputo differintegral is defined for $t\epsilon\langle a, T \rangle$ by formula:

$${}^c D_a^\alpha f(t) = I_a^{n_c - \alpha}\left[\frac{d^{n_c} f(t)}{dt^{n_c}}\right] \tag{11}$$

5 Illustrative Examples

We now consider the applications of the modified VIM to both ODEs and FDEs.

5.1 Ordinary differential equation

Example 1:

Consider the following differential equation:

$$\frac{du}{dt} + u = 0, \qquad u(0) = u_0,$$

This has the exact solution:

$$u(t) = u_0 e^{-t}$$

We can obtain the successive approximate solutions as:

$$u_{n+1}(t) = L^{-1}\left(\frac{u^{(m-1)}(0)}{s^m} + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L[u_n(t)]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_n]\right)$$

$$u_0(t) = u(0) = u_0$$

$$u_1(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_0]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s}L[1]\right) = L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s^2}\right)$$

$$= u_0 L^{-1}\left(\frac{1}{s} - \frac{1}{s^2}\right) = u_0(1 - t)$$

$$u_2(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_1]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L\big(u_0(1 - t)\big)\right) = u_0 L^{-1}\left(\frac{1}{s} - \frac{1}{s}L[1 - t]\right)$$

$$= u_0 L^{-1}\left(\frac{1}{s} - \frac{1}{s}\left(\frac{1}{s} - \frac{1}{s^2}\right)\right) = u_0 L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)$$

$$= u_0\left(1 - t + \frac{t^2}{2}\right)$$

$$= u_0 L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \cdots + (-1)^{n+1}\frac{1}{s^{n+1}}\right)$$

$$u_n(t) = u_0\left[\sum_{k=1}^{n}\frac{(-t)^n}{n!}\right]$$

For $n \to \infty$, $u_n(t)$ tends to the exact solution $u_0 e^{-t}$.

That the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler. Furthermore, the VIM can be easily extended to FDEs and this is the main purpose of this work

5.2 Fractional differential equations

Let us consider the FDE

$$\frac{du}{dt} + {}_0^\varsigma D_t^\alpha u = g(t,u),\ 0 < t,\ 0 < \alpha < 1,$$

Where $_0^cD_t^\alpha$ is Caputo differintegral denote by

and the variational iteration formula is given as

$$u_{n+1}(t) = u_n + \int_0^t \lambda(t,\tau)\left(\frac{du_n}{dt} + _0^cD_t^\alpha u_n - g(\tau, u_n)\right)d\tau,$$

Where $_0^cD_t^\alpha u$ is the Caputo derivative and $g(\tau, u_n)$ is a nonlinear term:

$$_0^cD_t^\alpha u + R[u] + N[u] = g(t)$$

$$u^{(k)}(0) = a_k, \ 0 < t, 0 < \alpha, m = [\alpha] + 1, k = 0, \dots, m - 1. \tag{12}$$

then, we consider the application of the modified VIM.

The following Laplace transform of the term $_0^cD_t^\alpha u$ holds

$$L[_0^cD_t^\alpha u] = s^\alpha u(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-1-k}, m - 1 < \alpha \leq m. \tag{13}$$

Taking the above Laplace transform to both sides of (11), the iteration formula of equation (11) can be constructed as:

$$u_{n+1}(s) = u_n(s) + \lambda(s) \times \left[s^\alpha u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]$$

Then consequently, after the identification of a Lagrange multiplier

$$\lambda(s) = -\frac{1}{s^\alpha},$$

then:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}\left[s^\alpha u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]\right]$$

$$= L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} - \frac{1}{s^\alpha}L\big(R[u_n] + N[u_n] - g(t)\big)\right),$$

$$m - 1 < \alpha \leq m \tag{14}$$

and

$$u_0(t) = L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1}\right)$$

$$= u(0) + u'(0) + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!} \tag{15}$$

Let us apply the above VIM to solve FDEs of Caputo type.

Example 2:

Consider the linear initial value problem

$$D^\alpha u + u = 0, \ u(0) = 1, u'(0) = 0, 0 < \alpha < 2. \tag{16}$$

After taking the Laplace transform to both sides of Equation (15) we get the following iteration formula:

$$u_{n+1}(s) = u_n(s) + \lambda(s^\alpha u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)]) \tag{17}$$

Setting $L[u_n(t)]$ as a restricted variation, $\lambda(s)$ can be identified as

$$\lambda(s) = -\frac{1}{s^\alpha} \tag{18}$$

The approximate solution of Equation (15) can be given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}\left(s^\alpha u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)]\right)\right]$$
$$= L^{-1}\left(\frac{1}{s^\alpha}\left(u(0)s^{\alpha-1} - L[u_n(t)]\right)\right)$$

$$u_{n+1}(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_n(t)]\right)$$

$$u_0(t) = 1$$

$$u_1(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_0(t)]\right) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[1]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}}\right) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_1(t)]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)$$

$$u_2(t) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_2(t)]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}} - \frac{1}{s^{3\alpha+1}}\right)$$

$$u_3(t) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Then $u_n(t)$ rapidly tends to the exact solution for $n \to \infty$:

$$u(t) = E_\alpha(-t)^\alpha$$

Since E_α is the Mittag-Leffler function defind as:

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}$$

5.3 Solution of equation (15) by using Laplace Transform

$$L[D^\alpha f(t)] = \frac{s^m F(s) - s^{m-1}F(0) - \cdots - F^{(m-1)}(0)}{s^{m-\alpha}}$$

$$m - 1 < \alpha \le m.$$

Then equation (15) will be:

$$\frac{s^2 u(s) - su(0)}{s^{2-\alpha}} + u(s) = 0$$

$$\frac{su(s)-1}{s^{1-\alpha}} + u(s) = 0$$

$$s^\alpha u(s) - s^{\alpha-1} + u(s) = 0$$

$$u(s)[s^\alpha + 1] = s^{\alpha-1}$$

$$u(s) = \frac{s^{\alpha-1}}{s^\alpha + 1} = \frac{1}{s}\left(\frac{1}{1+\frac{1}{s^\alpha}}\right)$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{-1}{s^\alpha}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{s^{n\alpha+1}}$$

By taking the inverse Laplace transform for both two sides give:

$$u(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty}\frac{(-t)^{n\alpha}}{\Gamma(n\alpha+1)} = E_\alpha(-t)^\alpha$$

6 Conclusion

Variational iteration method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this work, we have presented the modified variation iteration method which included Lagrange multiplier that easily identify by Laplace transform to give an analytical solutions of fractional differential equations, All examples showed that the results of the modified variational iteration method are in excellent agreement with those obtained by the Laplace transform method, but the results showed that the modified variational iteration method is more effective than the results of Laplace transform method because the inverse of Laplace transform some-times may be difficult to compute.

Competing Interests

Authors have declared that no competing interests exist.

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