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Research Article

Initial and Boundary Value Problems for a Class of Nonlinear Metaparabolic Equations

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This paper is devoted to the initial and boundary value problems for a class of nonlinear metaparabolic equations $u_t - \beta u_{xx} - k u_{xxt} + \gamma u_{xxxx} = f(u_x)_x$. At low initial energy level ($J(u_0) < d$), we not only prove the existence of global weak solutions for these problems by the combination of the Galerkin approximation and potential well methods but also obtain the finite time blow-up result by adopting the potential well and improved concavity skills. Finally, we also discussed the finite time blow-up phenomenon for certain solutions of these problems with high initial energy.

1. Introduction

In this paper, we study the initial and boundary value problems for the following nonlinear metaparabolic equations

$$u_t - \beta u_{xx} - k u_{xxt} + \gamma u_{xxxx} = f(u_x)_x, \quad x \in \Omega,\ t > 0, \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2}$$

$$u(0, t) = u(1, t) = 0, u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad t \geq 0, \tag{3}$$

in a bounded domain $\Omega = (0, 1)$, where $u_0(x)$ is the initial value function defined on Ω, $k > 0$ is the viscosity coefficient, $\gamma > 0$ is the interfacial energy parameter, and the nonlinear smooth function $f(s)$ satisfies the following assumptions:

$$\begin{cases} (i)|f(s)| \leq \alpha |s|^q, \quad \alpha > 0, 1 < q < +\infty, \forall s \in R, \\ (ii)(p+1)F(s) \geq sf(s) \text{ for some } p > 1, \quad \forall s \in R,\ F(s) = \int_0^s f(\tau)d\tau. \end{cases} \tag{4}$$

Equation (1) is a typical higher-order metaparabolic equation [1, 2], which has extensive physical background and rich theoretical connotation. This type of equation can be regarded as the regularization of Sobolev-Galpern equation by adding a fourth-order term u_{xxxx}. The Sobolev-Galpern equation appear in the study of various problems of fluid mechanics, solid mechanics, and heat conduction theory [3–5]. There have been many outstanding results about the qualitative theory for Sobolev-Galpern which include the existence, nonexistence, asymptotic behavior, regularities, and other some special properties of solutions. We also refer the reader to see [6, 7] and the papers cited therein. In (1), u is the concentration of one of the two phases, the fourth-order term γu_{xxxx} denotes the capillarity-driven surface diffusion, and the nonlinear term $f(u_x)_x$ is an intrinsic chemical potential. For example, differentiating (1) with respect to x and taking $v = u_x$, $\beta = 0$, then Equation (1) reduces to the well-known viscous Cahn-Hilliard equation

$$v_t - v_{xxt} + v_{xxxx} = \varphi(v)_{xx}, \quad x \in \Omega,\ t > 0. \tag{5}$$

Equation (5) appears in the dynamics of viscous first-order phase transitions in cooling binary solutions such as glasses, alloys, and polymer mixtures [8–10]. On the other hand, Equation (5) appears in the study of the regularization of nonclassical diffusion equations by adding a fourth-order term v_{xxxx}. There have been many outstanding results about the qualitative theory for this type of equations [11–15]. For example, Liu and Yin [13] studied Equation (5) for $\varphi(v) = -v + \gamma_1 v^2 + \gamma_2 v^3$ in R^3; they proved the existence and nonexistence of global classical solutions and pointed out that the sign of γ_2 is crucial to the global existence of solutions. In [14], Grinfeld and Novick-Cohen studied a Morse decomposition of the stationary solutions of the one-dimensional viscous Cahn-Hilliard equation by explicit energy calculations. They also proved a partial picture of the variation in the structure of the attractor ($n = 1$) for the viscous Cahn-Hilliard equation as the mass constraint and homotopy parameter are varied. Zhao and Liu [15] considered the initial boundary problem for the viscous Cahn-Hilliard Equation (5). In their paper, the optimal control under boundary condition was given, and the existence of optimal solution was proved.

Let us mention that there is an abounding literature about the initial and boundary value problems or Cauchy problem to nonlinear parabolic and hyperbolic equations. We refer the reader to the monographs [16, 17] which devoted to the second-order parabolic and pseudoparabolic problems. For the fourth-order nonlinear parabolic and hyperbolic equations, there are also some results about the initial boundary value and Cauchy problems, especially on global existence/nonexistence, uniqueness/nonuniqueness, and asymptotic behavior [18–25]. Bakiyevich and Shadrin [21] studied the Cauchy problem of the metaparabolic equation

$$\begin{cases} u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} = f(t,x), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

where $\alpha > 0$, $\beta \geq 0$, and $\gamma > 0$ are constants. They proved that the solutions are expressed through the sum of convolutions of functions $\varphi(x)$ and $f(t,x)$ with corresponding fundamental solutions.

In [22], Liu considered the metaparabolic equation

$$u_t - k u_{xxt} + A(u)_{xxxx} = f(x,t), \quad 0 \leq x \leq 1, 0 \leq t \leq T < +\infty, \tag{7}$$

where $A(u) = \int_0^u a(s)ds$, $a_0 + a_1|s|^b \leq a(s)$, and $|a''(s)| \leq a_2|s|^b$ (a_0, a_1, a_2, and b are positive constants). He proved the existence of weak solutions by using the method of continuity.

Khudaverdiyev and Farhadova [23] discussed the following fourth-order semilinear pseudoparabolic equation

$$u_t - \alpha u_{xxt} + u_{xxxx} = f(t,x,u,u_x,u_{xx},u_{xxx}), \quad 0 \leq x \leq 1, 0 \leq t \leq T < +\infty, \tag{8}$$

where $\alpha > 0$ is a fixed number. They proved the existence in large theorem (i.e., true for sufficiently large values of T) for generalized solution by means of Schauder stronger fixed-point principle.

In [24], Zhao and Xuan studied the generalized BBM-Burgers equation

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0. \tag{9}$$

They obtained the existence and convergence behavior of the global smooth solutions for Equation (9).

Philippin [25] studied the following fourth-order parabolic equation

$$u_t - k_1(t)\Delta u + k_2(t)\Delta^2 u = k_3(t)u|u|^{p-1}, \quad x \in \Omega, t > 0, \tag{10}$$

where k_i, $i = 1, 2, 3$ are positive constants or in general positive derivable functions of time t. Under appropriate assumptions on the data, he proved that the solutions u cannot exist for all time, and an upper bound is derived.

Equation (1) is also closely connected with many equations [26–29]. For example, Yang [26] considered the initial and boundary value problems of the following equation

$$u_{tt} + \lambda u_t + \alpha u_{xxxx} = f(u_x)_x, \quad x \in (0,1),\ t > 0. \tag{11}$$

He studied the asymptotic property of the solution and gave some sufficient conditions of the blow-up. When the weak damping term u_t of Equation (11) is replaced by the strong damping term $-u_{xxt}$, we have the following fourth-order wave equation

$$u_{tt} - 2b u_{xxt} + \alpha u_{xxxx} = f(u_x)_x, \quad x \in (0,1),\ t > 0. \tag{12}$$

Chen and Lu [27] studied the initial and boundary value problems of Equation (12). They proved the existence and uniqueness of the global generalized solution and global classical solution by the Galerkin method. Furthermore, Xu et al. [28] considered the initial and boundary value problems and proved the global existence and nonexistence of solutions by adopting and modifying the so called concavity method under some conditions with low initial energy. Ali Khelghati and Khadijeh Baghaei [29] proved that the blow up for Equation (12) occurs in finite time for arbitrary positive initial energy.

Motivated by the above researches, in the present work, we mainly study the initial and boundary value problems (1)–(3) of metaparabolic equations. Hereafter, for simplicity, we set $\alpha = \beta = \gamma = 1$. Especially, the appearance of the dispersion term u_{xxt} and nonlinearity $f(u_x)_x$ for these problems cause some difficulties such that we cannot apply the normal Galerkin approximation, concavity, and potential methods directly; we have to invent some new skills and methods to overcome these difficulties.

Our paper is organized as follows. In Section 2, we introduce some functionals and potential wells and discuss the invariance of some sets which are needed for our work. In Sections 3 and 4, the existence and nonexistence of global weak solutions for problems (1)–(3) are proved by the Galerkin approximation and potential well and improved concavity methods at low initial energy ($J(u_0) < d$). Especially, the threshold result between global existence and nonexistence is obtained under certain conditions. In the last section, we investigate the finite time blow-up for certain solutions of problems (1)–(3) with high initial energy.

2. Preliminaries

In this section, we introduce some functionals, potential wells, and important lemmas that will be needed in this paper. Throughout this paper, the following abbreviations are used for precise statement:

$$\begin{gathered} L^q(\Omega) = L^q,\ W^{m,q}(\Omega) = W^{m,q}, \\ H^m(\Omega) = W^{m,2}(\Omega) = H^m,\ H_0^m(\Omega) = H_0^m, \\ D = \{u \in H^2(\Omega) \mid u(0,t) = u(1,t) \\ = 0,\ u_{xx}(0,t) = u_{xx}(1,t) = 0\}, \\ \|u\|_{L^q(\Omega)} = \|u\|_q,\ \|u\|_{W^{m,q}(\Omega)} \\ = \|u\|_{W^{m,q}},\ \|u\|_{H_0^m(\Omega)} = \|u\|_{H_0^m}. \end{gathered} \tag{13}$$

And the notation $(\cdot, \cdot)$ for the L^2-inner product will also be used for the notation of duality paring between dual spaces.

First of all, let us consider the functionals as follows. The "total energy" and "potential energy" associated with the problems (1)–(3) are defined by

$$\begin{gathered} E(t) := E(u(t)) = \int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau + \frac{1}{2}\|u_x\|_{H^1}^2 + \int_\Omega F(u_x)dx, \\ J(u) = \frac{1}{2}\|u_x\|_{H^1}^2 + \int_\Omega F(u_x)dx, \\ I(u) = \|u_x\|_{H^1}^2 + \int_\Omega f(u_x)u_x dx. \end{gathered} \tag{14}$$

Then, by simple calculation, it follows that

$$J'(u) = \frac{d}{dt}J(u) = -\|u_t(t)\|_{H^1}^2 \le 0, \tag{15}$$

$$E(t) = J(u) + \int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau = J(u_0) = E(0). \tag{16}$$

The corresponding "Nehari manifold" and "potential well depth" are given by

$$\begin{gathered} N = \{u \in D \cap W^{1,q+1}(\Omega) \mid I(u) = 0, u \ne 0\}, \\ d = \inf_{u\in N}(J(u)). \end{gathered} \tag{17}$$

In addition, we define

$$\begin{gathered} N^+ = \{u \in D \cap W^{1,q+1}(\Omega) \mid I(u) > 0\} \cup \{0\}, \\ N^- = \{u \in D \cap W^{1,q+1}(\Omega) \mid I(u) < 0\}. \end{gathered} \tag{18}$$

To obtain the results of this paper, we also introduce so called stable and unstable sets:

$$\begin{gathered} W = \{u \in D \cap W^{1,q+1}(\Omega) \mid I(u) > 0, J(u) < d\} \cup \{0\}, \\ V = \{u \in D \cap W^{1,q+1}(\Omega) \mid I(u) < 0, J(u) < d\}. \end{gathered} \tag{19}$$

Next, we shall give the following some essential lemmas which are important to obtain the main results of this paper.

Lemma 1. *Let $f(s)$ satisfy (4), $u \in D \cap W^{1,q+1}(\Omega)$, then the following hold:*

(1) If $0 < \|u_x\|_{H^1} < \gamma_0$, then $u \in N^+ (u \ne 0)$;

(2) If $u \in N^-$, then $\|u_x\|_{H^1} > \gamma_0$

(3) If $u \in N$, then $\|u_x\|_{H^1} \ge \gamma_0$, where

$$\begin{gathered} \gamma_0 = \left(\frac{1}{\alpha C_*^{q+1}}\right)^{1/(q-1)}, \\ C_* = \sup \frac{\|u_x\|_{q+1}}{\|u_x\|_{H^1}}, \\ \left(u \in D \cap W^{1,q+1}(\Omega), u \ne 0\right). \end{gathered} \tag{20}$$

Proof.

(1) If $0 < \|u_x\|_{H^1} < \gamma_0$, then

$$\begin{aligned} -\int_\Omega f(u_x)u_x dx &\le \int_\Omega |f(u_x)u_x| dx \le \alpha \int_\Omega |u_x|^{q+1} dx \\ &\le \alpha C_*^{q+1}\|u_x\|_{H^1}^{q+1} = \alpha C_*^{q+1}\|u_x\|_{H^1}^{q-1}\|u_x\|_{H^1}^2 \\ &< \|u_x\|_{H^1}^2, \end{aligned} \tag{21}$$

which gives $I(u) > 0$ or $u \in N^+ (u \ne 0)$.

(2) If $u \in N^-$, then $u \ne 0$ and

$$\|u_x\|_{H^1}^2 < -\int_\Omega f(u_x)u_x dx \le \alpha C_*^{q+1}\|u_x\|_{H^1}^{q-1}\|u_x\|_{H^1}^2, \tag{22}$$

which gives $\|u_x\|_{H^1} > \gamma_0$.

(3) If $u \in N$, then from

$$\|u_x\|_{H^1}^2 = -\int_\Omega f(u_x)u_x dx \le \alpha C_*^{q+1}\|u_x\|_{H^1}^{q-1}\|u_x\|_{H^1}^2, \tag{23}$$

we have $\|u_x\|_{H^1} \ge \gamma_0$.

Lemma 2. *Let $f(s)$ satisfy (4) and $u \in N$, then*

$$d \ge d_0 = \frac{p-1}{2(p+1)}\gamma_0^2 = \frac{p-1}{2(p+1)}\left(\frac{1}{\alpha C_*^{q+1}}\right)^{2/(q-1)}. \tag{24}$$

Proof. For any $u \in N$, we have by Lemma 1 (3) that $\|u_x\|_{H^1} \geq \gamma_0$. Hence, from

$$\begin{aligned} J(u) &= \frac{1}{2}\|u_x\|_{H^1}^2 + \int_\Omega F(u_x)dx \\ &\geq \frac{1}{2}\|u_x\|_{H^1}^2 + \frac{1}{p+1}\int_\Omega f(u_x)u_x dx \\ &= \frac{p-1}{2(p+1)}\|u_x\|_{H^1}^2 + \frac{1}{p+1}I(u) \\ &= \frac{p-1}{2(p+1)}\|u_x\|_{H^1}^2 \geq \frac{p-1}{2(p+1)}\gamma_0^2, \end{aligned} \tag{25}$$

and the definition of potential depth d, we get $d \geq d_0$.

For simplicity, we define the weak solution of (1)–(3) over the interval $\Omega \times [0, T)$, but it is to be understood that T is either infinity or the limit of the existence interval.

Definition 3. We say that $u(x, t)$ is called a weak solution of the problems (1)–(3) on the interval $\Omega \times [0, T)$. If $u \in L^\infty([0, T]; D \cap W^{1,q+1}(\Omega))$, with $u_t \in L^2([0, T]; H_0^1(\Omega))$ satisfy the following conditions

(i) For any $v \in D \cap W^{1,q+1}(\Omega)$, such that

$$(u_t, v) + (u_{xt}, v_x) + (u_x, v_x) + (u_{xx}, v_{xx}) = -(f(u_x), v_x). \tag{26}$$

(ii) $u(x, 0) = u_0$ in $D \cap W^{1,q+1}(\Omega)$.

(iii) The following energy inequality holds

$$J(u) \leq J(u_0), \tag{27}$$

for any $0 \leq t < T$.

Lemma 4. *Let $f(s)$ satisfy (4) and $u(x, t)$ be a solution of (1)–(3) over the interval $[0, T)$. If there exists a time $t_0 \in [0, T)$ such that $u(t_0) \in W$, then $u(t) \in W$ for any $t \in [t_0, T)$, where T is either infinity or the limit of the existence interval.*

Proof. Arguing by contradiction and considering the time continuity of $I(u)$ and $J(u)$, we suppose that there exists a time $t_1 \in [t_0, T)$ such that $u(x, t) \in W$ for any $t \in [t_0, t_1)$, but $u(t_1) \in \partial W$, which means that (1) $J(u(t_1)) = d$ or (2) $I(u(t_1)) = 0$, $\|u(t_1)\|_{H^2} \neq 0$. By (15) and $u(t_0) \in W$, we have $J(u(t_1)) \leq J(u(t_0)) < d$. It follows that case (1) is impossible. If $I(u(t_1)) = 0$, $\|u(t_1)\|_{H^2} \neq 0$, then by the definition of d, we have $J(u(t_1)) > d$ which contradicts $J(u(t_1)) \leq J(u(t_0)) < d$. The case (2) is also impossible.

Lemma 5. *Let $f(s)$ satisfy (4) and $u(x, t)$ be a solution of (1)–(3) over the interval $[0, T)$. If there exists a time $t_0 \in [0, T)$ such that $u(t_0) \in V$, then $u(t) \in V$ for any $t \in [t_0, T)$, where T is either infinity or the limit of the existence interval.*

Proof. The proof of Lemma 5 is similar to Lemma 4.

Lemma 6 (see [29, 30]). *Assume that the function $\phi(t) \in C^2$, $\phi(t) \geq 0$ satisfies*

$$\phi(t)\phi''(t) - (1+\delta)\phi'^2(t) \geq 0, \tag{28}$$

for certain real number $\delta > 0$, $\phi(0) > 0$, and $\phi'(0) > 0$. Then, there exists a real number $\tilde{T}$ with $0 < \tilde{T} \leq (\phi(0))/(\alpha\phi'(0))$ such that

$$\phi(t) \longrightarrow \infty, \quad ast \longrightarrow \tilde{T}^-. \tag{29}$$

We construct an approximate weak solution of the problems (1)-(3) by the Galerkin¡¯s method. Let $\{w_j\}_{j=1}^\infty$ be the eigenfunction system of problem

$$-w_{jxx} = \lambda w_j, \text{ in } \Omega, w_j(0) = w_j(1) = 0, j = 1, 2, \cdots. \tag{30}$$

Obviously, there exist some basis such that $\{w_j\}_{j=1}^\infty \subseteq D \cap W^{1,q+1}(\Omega)$, and it is dense in $D \cap W^{1,q+1}(\Omega)$. Now, suppose that the approximate weak solution of the problems (1)–(3) can be written

$$u_m(x, t) = \sum_{j=1}^m d_m^j(t) w_j(x). \tag{31}$$

According to Galerkin's method, these coefficients $d_m^j(t)$ need to satisfy the following initial value problem of the nonlinear differential equations

$$\begin{cases} (u_{mt}, w_j) + (u_{mxt}, w_{jx}) + (u_{mx}, w_{jx}) + (u_{mxx}, w_{jxx}) = -(f(\mathrm{u}_{mx}), w_{jx}), \\ u_m|_{t=0} = u_{0m}(x), \end{cases} \tag{32}$$

where $u_{0m}(x) = \sum_{j=1}^m d_m^j(0) w_j(x)$, $u_{0m}(x) \longrightarrow u_0(x)$, in $D \cap W^{1,q+1}(\Omega)$.

The initial value problem (32) possesses a local solution in $[0, t_m)$, $0 < t_m < T$ for an arbitrary $T > 0$. Under some appropriate assumptions on the nonlinear terms and the initial data, we prove that the system (32) has global weak solutions in the interval $[0, T]$. Furthermore, we show that the solutions of the problems (1)–(3) can be approximated by the functions $u_m(x, t)$.

3. Existence of Global Weak Solutions

In this section, we shall prove the existence of global weak solution by the combination of the Galerkin approximation and potential well methods.

Theorem 7. *Assume that f satisfy (4), and $u_0 \in W$, then the problems (1)–(3) admits a global weak solution $u \in L^\infty([0, \infty); D \cap W^{1,q+1}(\Omega))$, with $u_t \in L^2([0, \infty); H_0^1(\Omega))$ and $u \in W$ for all $0 \leq t < \infty$.*

Proof. Multiplying (32) by $d_m^j(t)'$ and summing for $j=1,\cdots,m$, then we have

$$\begin{aligned}&(u_{mt},u_{mt})+(u_{mx},u_{mxt})+(u_{mxt},u_{mxt})+(u_{mxx},u_{mxxt})\\&\quad=-(f(u_{mx}),u_{mxt}).\end{aligned}\tag{33}$$

By a direct calculation, it follows that

$$\int_0^t\|u_m(\tau)\|_{H^1}^2d\tau+J(u_m)=J(u_{0m}),\tag{34}$$

where

$$J(u_m)=\frac{1}{2}\|u_{mx}\|_{H^1}^2+\int_\Omega F(u_{mx})dx.\tag{35}$$

Utilizing the strong convergence of u_{0m} in $D\cap W^{1,q+1}(\Omega)$, we note that $J(u_{0m})\longrightarrow J(u_0)<d$. Hence, we get $J(u_{0m})<d$ for sufficiently large m. On the other hand, from $u_0\in W$ and $u_{0m}(x)\longrightarrow u_0(x)$ in $D\cap W^{1,q+1}(\Omega)$, it follows that $u_{0m}\in W$ for sufficiently large m. Similar to the proof of Lemma 4, we have that the solution u_m constructed by (31) remains in W for $0\le t<\infty$ and sufficiently large m.

Thus, from (4) and

$$\begin{aligned}d>J(u_m)&=\frac{1}{2}\|u_{mx}\|_{H^1}^2+\int_\Omega F(u_{mx})dx\\&\ge\frac{1}{2}\|u_{mx}\|_{H^1}^2+\frac{1}{p+1}\int_\Omega f(u_{mx})u_{mx}dx\\&\ge\frac{p-1}{2(p+1)}\|u_{mx}\|_{H^1}^2+\frac{1}{p+1}I(u_m)\ge0,\end{aligned}\tag{36}$$

we obtain

$$\begin{aligned}&\int_0^t\|u_m(\tau)\|_{H^1}^2d\tau<d,\quad 0\le t<\infty,\\&\|u_{mx}\|_{H^1}^2<\frac{2(p+1)}{p-1}d,\quad 0\le t<\infty,\\&\|u_{mx}\|_{q+1}^2\le C_*^2\|u_{mx}\|_{H^1}^2<C_*^2\frac{2(p+1)}{p-1}d,\quad 0\le t<\infty,\\&\|f(u_{mx})\|_r^r\le\alpha^r\|u_{mx}\|_{r+1}^{r+1}\\&\qquad<\alpha^rC_*^{r+1}\left(\frac{2(p+1)}{p-1}d\right)^{(r+1)/2},\quad 0\le t<\infty,\end{aligned}\tag{37}$$

where $r=(q+1)/q$, $1/(q+1)+(1/r)=1$. Therefore, there exist a subsequence of $\{u_m\}$ which from now on will be also denoted by $\{u_m\}$ such that as $m\longrightarrow\infty$

$$u_m\longrightarrow u\text{ in }L^\infty\left([0,\infty);D\cap W^{1,q+1}(\Omega)\right)\text{ weakly star,}\tag{38}$$

$$u_m\longrightarrow u\ a.e.Q=\Omega\times[0,\infty),\tag{39}$$

$$u_{mt}\longrightarrow u_t\text{ in }L^2\left([0,\infty);H_0^1(\Omega)\right)\text{weakly,}\tag{40}$$

$$u_{mx}\longrightarrow u_x\text{ in }L^\infty\left([0,\infty);L^{q+1}(\Omega)\right)\text{weakly star,}\tag{41}$$

$$f(u_{mx})\longrightarrow\chi=f(u_x)\text{ in }L^\infty([0,\infty);L^r(\Omega))\text{weakly star.}\tag{42}$$

Convergences (38)–(42) permit us to pass to the limit in (32). Taking $m\longrightarrow\infty$, we obtain

$$\left(u_t,w_j\right)+\left(u_{xt},w_{jx}\right)+\left(u_x,w_{jx}\right)+\left(u_{xx},w_{jxx}\right)=-\left(f(u_x),w_{jx}\right),\tag{43}$$

for $j=1,2\cdots$. Considering that the basis $\{w_j\}_{j=1}^\infty$ are dense in $D\cap W^{1,q+1}(\Omega)$, we choose a function $v\in L^\infty([0,\infty);D\cap W^{1,q+1}(\Omega))$ having the form $v(t)=\sum_{j=1}^\infty d^j(t)w_j$, where $\{d^j(t)\}_{j=1}^\infty$ are given functions. Multiplying (43) by $d^j(t)$ and summing $j=1,2,\cdots$, then we have

$$(u_t,v)+(u_{xt},v_x)+(u_x,v_x)+(u_{xx},v_{xx})=-(f(u_x),v_x).\tag{44}$$

Moreover, (32) gives $u(x,0)=u_0(x)$ in $D\cap W^{1,q+1}(\Omega)$. Next, we will prove that u satisfies (27). Taking into account the nonlinear term of the functional $J(u)$, we deduce

$$\begin{aligned}&\left|\int_\Omega F(u_{mx})dx-\int_\Omega F(u_x)dx\right|\\&\quad=\left|\int_\Omega f(u_x+\theta_mu_{mx})(u_{mx}-u_x)dx\right|\\&\quad\le\|f(u_x+\theta_mu_{mx})\|_{q+1/q}\|u_{mx}-u\|_{q+1}\\&\quad\le C\|u_{mx}-u_x\|_{q+1}\longrightarrow0,\end{aligned}\tag{45}$$

as $m\longrightarrow\infty$, where $0<\theta_m<1$. Hence, we have

$$\lim_{m\longrightarrow\infty}\int_\Omega F(u_{mx})dx=\int_\Omega F(u_x)dx.\tag{46}$$

Then, making use of Fatou's Lemma and (34), (46), we deduce

$$\begin{aligned}\frac{1}{2}\|u_x\|_{H^1}^2&\le\liminf_{m\longrightarrow\infty}\frac{1}{2}\|u_{mx}\|_{H^1}^2=\liminf_{m\longrightarrow\infty}\left[J(u_m)-\int_\Omega F(u_{mx})dx\right]\\&\le\lim_{m\longrightarrow\infty}\left[J(u_{0m})-\int_\Omega F(u_{mx})dx\right]\\&=J(u_0)-\int_\Omega F(u_x)dx,\end{aligned}\tag{47}$$

which yields (27). Thus, we obtain that u is a global weak solution of problems (1)–(3). Finally, making use of Lemma 4 again, we get $u(t)\in W$ for $0\le t<\infty$.

4. Finite Time Blow-up of Solutions with $J(u_0) < d_0$

In this section, we consider the finite time blow up of solutions with $E(0) = J(u_0) < d_0$ for the problems (1)–(3).

Theorem 8. *Let f satisfy (4), and $u_0 \in D \cap W^{1,q+1}(\Omega)$. Assume that $I(u_0) < 0$ and $E(0) = J(u_0) < d_0$, where d_0 is defined in Lemma 2, then the weak solution $u(t)$ of problems (1)–(3) blow-up in finite time.*

Proof. Let $u(t)$ be any weak solution of the problems (1)–(3) with $I(u_0) < 0$ and $E(0) = J(u_0) < d_0$, $\tilde{T}$ be the maximal existence time of $u(t)$. Next, we will prove $\tilde{T} < +\infty$. Arguing by contradiction, we suppose $\tilde{T} = +\infty$. We define the function $\Psi : [0, T_1] \longrightarrow R^+$ by

$$\Psi(t) = \int_0^t \|u(\tau)\|_{H^1}^2 d\tau + (T_1 - t)\|u_0\|_{H^1}^2 + b(t + T_0)^2, \quad (48)$$

where b, T_0, and T_1 are positive constants to be chosen later. By simple calculation, we have

$$\begin{aligned}\Psi'(t) &= \|u(t)\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2b(t + T_0)\\ &= 2\int_0^t \int_\Omega u(\tau)u_t(\tau)dxd\tau\\ &\quad + 2\int_0^t \int_\Omega u_x(\tau)u_{xt}(\tau)dxd\tau + 2b(t + T_0),\\ \Psi''(t) &= 2\int_\Omega u(t)u_t(t)dx + 2\int_\Omega u_x(t)u_{xt}(t)dx + 2b.\end{aligned} \quad (49)$$

By (1), we obtain

$$\begin{aligned}\Psi''(t) &= 2\int_\Omega u\left[u_{xx} - u_{xxxx} + f(u_x)_x\right]dx + 2b\\ &= -2\|u_x\|_{H^1}^2 - 2\int_\Omega u_x f(u_x)dx + 2b.\end{aligned} \quad (50)$$

Therefore, we can get

$$\begin{aligned}&\Psi(t)\Psi''(t) - \frac{p+3}{4}\Psi'(t)^2\\ &= \Psi(t)\left[-2\|u_x\|_{H^1}^2 - 2\int_\Omega u_x f(u_x)dx + 2b\right]\\ &\quad - (p+3)\left[\int_0^t \int_\Omega u(\tau)u_t(\tau)dxd\tau\right.\\ &\quad \left. + \int_0^t \int_\Omega u_x(\tau)u_{xt}(\tau)dxd\tau + b(t + T_0)\right]^2\\ &= \Psi(t)\left[-2\|u_x\|_{H^1}^2 - 2\int_\Omega u_x f(u_x)dx + 2b\right]\\ &\quad + (p+3)\left\{\eta(t) - \left[\Psi(t) - (T_1 - t)\|u_0\|_{H^1}^2\right]\right.\\ &\quad \left.\cdot \left[\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau + b\right]\right\},\end{aligned} \quad (51)$$

where

$$\begin{aligned}\eta(t) &= \left[\int_0^t \|u(\tau)\|_{H^1}^2 d\tau + b(t + T_0)^2\right]\left[\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau + b\right]\\ &\quad - \left[\int_0^t \int_\Omega u(\tau)u_t(\tau)dxd\tau + \int_0^t \int_\Omega u_x(\tau)u_{xt}(\tau)dxd\tau\right.\\ &\quad \left. + b(t + T_0)\right]^2.\end{aligned} \quad (52)$$

Using the Schwarz and Young inequalities, we have

$$\left(\int_0^t (u(\tau), u_t(\tau))d\tau\right)^2 \le \int_0^t \|u(\tau)\|_2^2 d\tau \int_0^t \|u_t(\tau)\|_2^2 d\tau, \quad (53)$$

$$\left(\int_0^t (u_x(\tau), u_{xt}(\tau))d\tau\right)^2 \le \int_0^t \|u_x(\tau)\|_2^2 d\tau \int_0^t \|u_{xt}(\tau)\|_2^2 d\tau, \quad (54)$$

$$\begin{aligned}&\int_0^t (u(\tau), u_t(\tau))d\tau \int_0^t (u_x(\tau), u_{xt}(\tau))d\tau\\ &\le \frac{1}{2}\int_0^t \|u(\tau)\|_2^2 d\tau \int_0^t \|u_{xt}(\tau)\|_2^2 d\tau + \frac{1}{2}\int_0^t \|u_t\|_2^2 d\tau \int_0^t \|\nabla u\|_2^2 d\tau.\end{aligned} \quad (55)$$

Inserting (53)–(55) into (52), we have

$$\eta(t) \ge 0, \quad t \in [0, T_1]. \quad (56)$$

Thus,

$$\Psi''(t)\Psi(t) - \frac{p+3}{4}\Psi'(t)^2 \ge \Psi(t)\xi(t), \quad (57)$$

where

$$\begin{aligned}\xi(t) &= -2\|u_x\|_{H^1}^2 - 2\int_\Omega u_x f(u_x)dx + 2b\\ &\quad - (p+3)\left[\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau + b\right]\\ &\ge -(p+3)\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau - (p+1)b\\ &\quad - 2(p+1)\int_\Omega F(u_x)dx - 2\|u_x\|_{H^1}^2\\ &= -2(p+1)\left[E(t) - \frac{1}{2}\|u_x\|_{H^1}^2 - \int_0^t \|u_t(\tau)\|_{H^1} d\tau\right]\\ &\quad - (p+3)\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau - (p+1)b - 2\|u_x\|_{H^1}^2\\ &\ge -2(p+1)E(0) + (p-1)\|u_x\|_{H^1}^2\\ &\quad + (p-1)\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau - (p+1)b.\end{aligned} \quad (58)$$

From $I(u_0) < 0$, $E(0) = J(u_0) < d_0$ and Lemma 5, we have $I(u) < 0$ for all $t \in [0,\infty)$. Hence, by Lemma 1, it follows that

$\|u_x\|_{H^1}^2 > \gamma_0^2 = ((2(p+1))/(p-1))d_0 = (1/(\alpha C_*^{p+1}))^{2/(p-1)}$. Thus, we have from (58) that

$$\xi(t) > 2(p+1)(d_0 - J(u_0)) - (p+1)b. \tag{59}$$

We choose b small enough such that $b \le 2(d_0 - J(u_0))$, then we have

$$\Psi''(t)\Psi(t) - \frac{p+3}{4}\Psi'(t)^2 \ge \Psi(t)\xi(t) > 0, \tag{60}$$

for all $t \in [0, T_1]$. From $I(u) < 0$ for all $t \in [0, T_1)$ and (50), we get $\Psi''(t) > 0$. Hence, we have $\Psi'(t) > \Psi'(0) = 2bT_0 > 0$ for all $t \in (0, T_1)$.

From what has been discussed above, using Lemma 6 and letting $\delta = (p-1)/4$, we can obtain that there exists a finite time $\tilde{T} > 0$ such that

$$\lim_{t \longrightarrow \tilde{T}^-} \Psi(t) = +\infty, \tag{61}$$

or

$$\lim_{t \longrightarrow \tilde{T}^-} \left(\int_0^t \|u(\tau)\|_{H^1}^2 d\tau + (T_1 - t)\|u_0\|_{H^1}^2 + b(t + T_0)^2 \right) = +\infty, \tag{62}$$

where

$$\tilde{T} \le \frac{\Psi(0)}{\gamma \Psi'(0)} = \frac{2\left[T_1\|u_0\|_{H^1}^2 + bT_0^2\right]}{(p-1)bT_0}. \tag{63}$$

which contradicts $\tilde{T} = +\infty$. Hence, the desired assertion immediately follows.

From the discussed above in Sections 3 and 4, a threshold result of global existence and nonexistence of solutions for problems (1)–(3) has been obtained as follows.

Corollary 9. *Let f satisfy (4), and $u_0 \in D \cap W^{1,q+1}(\Omega)$. Assume that $E(0) = J(u_0) < d_0$. Then, problems (1)–(3) admits a global weak solution provided $I(u_0) \ge 0$ (includes $u_0 = 0$); Problems (1)–(3) dose not admit any global solution provided $I(u_0) < 0$.*

5. Finite Time Blow-up of Solutions with High Initial Energy

In this section, we shall state and prove the finite time blow-up result with high initial energy for the problems (1)–(3).

Theorem 10. *Let f satisfy (4), and $u_0 \in D \cap W^{1,q+1}(\Omega)$. Assume that*

$$0 \le J(u_0) \le \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H^1}^2, \tag{64}$$

where λ_1 is the optimal constant satisfying the Poincáre inequality $\|u_x\|_2^2 \ge \lambda_1\|u\|_2^2$, then the weak solution $u(t)$ of problems (1)–(3) blow-up in finite time.

Proof. Arguing by contradiction, we suppose that $u(t)$ is a global weak solution of the problems (1)–(3). Considering that

$$\int_0^t u_t(\tau)d\tau = u(t) - u_0, \quad \forall t \in [0,\infty), \tag{65}$$

so we have

$$\begin{aligned} \int_0^t \|u_t(\tau)\|_{H^1} d\tau &\ge \left\|\int_0^t u_t(\tau)d\tau\right\|_{H^1} = \|u(t) - u_0\|_{H^1} \\ &\ge \|u(t)\|_{H^1} - \|u_0\|_{H^1}. \end{aligned} \tag{66}$$

From (16), (66) and Hölder's inequality, we obtain

$$\begin{aligned} \|u(t)\|_{H^1} &\le \|u_0\|_{H^1} + \int_0^t \|u_t(\tau)\|_{H^1} d\tau \\ &\le \|u_0\|_{H^1} + t^{1/2}\left(\int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau\right)^{1/2} \\ &\le \|u_0\|_{H^1} + t^{1/2}(J(u_0) - J(u(t)))^{1/2}. \end{aligned} \tag{67}$$

Since assume that $u(t)$ is a global weak solution of the problems (1)–(3), we get $J(u(t)) \ge 0$ for all $t \in [0, \infty)$. Otherwise, there exists a time $t_0 \in [0,\infty)$ such that $J(u(t_0)) < 0$. Hence, from

$$\begin{aligned} 0 > J(u(t_0)) &= \frac{1}{2}\|u_x(t_0)\|_{H^1}^2 + \int_\Omega F(u_x(t_0))dx \\ &\ge \frac{1}{2}\|u_x(t_0)\|_{H^1}^2 + \frac{1}{p+1}\int_\Omega f(u_x(t_0))u_x(t_0)dx \\ &\ge \frac{p-1}{2(p+1)}\|u_x(t_0)\|_{H^1}^2 + \frac{1}{p+1}I(u(t_0)), \end{aligned} \tag{68}$$

we have $I(u(t_0)) < 0$ and $J(u(t_0)) < 0$ which implies that $u(t_0) \in V$. Therefore, by the results of Theorem 8, we obtain that $u(x,t;u(t_0)) = u(x, t - t_0; u_0)$ blows up in finite time, which is a contradiction. Thus, we have

$$J(u_0) \ge J(u(t)) \ge 0, \quad \text{for all } t \in [0,\infty). \tag{69}$$

Next, combining (67) and (69), we get

$$\begin{aligned} \|u(t)\|_{H^1} &\le \|u_0\|_{H^1} + t^{1/2}(J(u_0) - J(u(t)))^{1/2} \\ &\le \|u_0\|_{H^1} + t^{1/2}(J(u_0))^{1/2}, \end{aligned} \tag{70}$$

for all $t \in [0, \infty)$.

On the other hand, multiplying u on two sides of Equation (1) and integrating by parts, we have

$$(u_t, u) + (u_{xt}, u_x) + (u_x, u_x) + (u_{xx}, u_{xx}) = -(f(u_x), u_x). \tag{71}$$

The Poinca're inequality gives $\|u_x\|_2^2 \geq \lambda_1 \|u\|_2^2$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} w_{xx} + \lambda w = 0, \text{ in } \Omega, \\ w = 0, \text{ on } \partial\Omega. \end{cases} \tag{72}$$

Thus, we have

$$\|u\|_{H^1}^2 = \|u\|_2^2 + \|u_x\|_2^2 \leq \frac{1+\lambda_1}{\lambda_1}\|u_x\|_2^2 \leq \frac{1+\lambda_1}{\lambda_1}\|u_x\|_{H^1}^2. \tag{73}$$

By the combination of (4), (73), and Sobolev's inequality, we can get that

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u\|_{H^1}^2\right) &= -\|u_x\|_{H^1}^2 - (f(u_x), u_x) \\ &\geq -\|u_x\|_{H^1}^2 - (p+1)\int_\Omega F(u_x)dx \\ &= \frac{p-1}{2}\|u_x\|_{H^1}^2 - (p+1)J(u(t)) \\ &\geq \frac{(p-1)\lambda_1}{2(1+\lambda_1)}\|u\|_{H^1}^2 - (p+1)J(u(t)). \end{aligned} \tag{74}$$

Since $(d/dt)J(u(t)) \leq 0$, for $\forall k > 0$, we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u\|_{H^1}^2 - kJ(u(t))\right) \\ &\quad\geq \frac{d}{dt}\left(\frac{1}{2}\|u\|_{H^1}^2\right) \geq \frac{(p-1)\lambda_1}{2(1+\lambda_1)}\|u\|_{H^1}^2 - (p+1)J(u(t)), \\ &\quad= \frac{(p-1)\lambda_1}{1+\lambda_1}\left[\frac{1}{2}\|u\|_{H^1}^2 - \frac{(p+1)(1+\lambda_1)}{(p-1)\lambda_1}J(u(t))\right]. \end{aligned} \tag{75}$$

Taking $k = ((p+1)(1+\lambda_1))/((p-1)\lambda_1)$ in (75) and $G(t) = 1/2\|u\|_{H^1}^2 - (((p+1)(1+\lambda_1))/((p-1)\lambda_1))J(u(t))$, then we have

$$\frac{d}{dt}G(t) \geq \frac{(p-1)\lambda_1}{1+\lambda_1}G(t). \tag{76}$$

Integrating the inequality (76) from 0 to t, we see

$$G(t) \geq e^{(((p-1)\lambda_1)/(1+\lambda_1))t}G(0), \quad t \in [0,\infty), \tag{77}$$

which means that

$$\begin{aligned} \|u\|_{H^1}^2 \geq{}& \frac{2(p+1)\lambda_1}{(p-1)(1+\lambda_1)}J(u(t)) \\ &+ 2e^{(((p-1)\lambda_1)/(1+\lambda_1))t}G(0), \quad t \in [0,\infty). \end{aligned} \tag{78}$$

From the assumption condition (64), we have $G(0) > 0$. Hence, we get from (69) and (78) that $\|u\|_{H^1}^2 \geq 2e^{(((p-1)\lambda_1)/(1+\lambda_1))t}G(0)$, i.e.,

$$\|u\|_{H^1} \geq [2G(0)]^{1/2}e^{(((p-1)\lambda_1)/(2(1+\lambda_1)))t}, \quad t \in [0,\infty). \tag{79}$$

From the combination of (70) and (79), we have

$$[2G(0)]^{1/2}e^{(((p-1)\lambda_1)/(2(1+\lambda_1)))t} \leq \|u_0\|_{H^1} + t^{1/2}(J(u_0))^{1/2}. \tag{80}$$

Clearly, the above inequality cannot hold for t large enough, this means that the solution u of problems (1)–(3) cannot exist all time.

Furthermore, by (80) and $e^{(((p-1)\lambda_1)/(2(1+\lambda_1)))t} \geq (((p-1)\lambda_1)/(2(1+\lambda_1)))t$, we can obtain the inequality

$$\frac{\sqrt{2G(0)}(p-1)\lambda_1}{2(1+\lambda_1)}t - (J(u_0))^{1/2}t^{1/2} - \|u_0\|_{H^1} \leq 0, \tag{81}$$

which implies that there exists a finite time $\tilde{T} > 0$ such that

$$\lim_{t \longrightarrow \tilde{T}^-}\|u\|_{H^1}^2 = +\infty, \tag{82}$$

and $\tilde{T}^{1/2}$ is the largest root of the following equation

$$\frac{\sqrt{2G(0)}(p-1)\lambda_1}{2(1+\lambda_1)}t^2 - (J(u_0))^{1/2}t - \|u_0\|_{H^1} = 0. \tag{83}$$

This completes the proof.

6. Conclusion and Future Work

In our work, we mainly study the qualitative properties of the solutions for the initial and boundary value problems (1)–(3). It is well known that Equation (1) is a typical higher-order metaparabolic equation, which has extensive practical background and rich theoretical connotation. For example, the solutions u of (1) can be used to denote the concentration of one of the two phases, the fourth-order term γu_{xxxx} presents the capillarity-driven surface diffusion, and the nonlinear term $f(u_x)_x$ is an intrinsic chemical potential. Especially, the interaction between the dispersion term u_{xxt} and nonlinearity $f(u_x)_x$ of these problems cause some difficulties such that we cannot apply the normal Galerkin approximation, concavity, and potential methods directly. Considering the above situation, at low initial energy level, we first prove the existence of global weak solutions for these problems by the Galerkin approximation and potential well methods and obtain the finite time blow-up result by the potential well and improved concavity skills. In addition, we establish the finite time blow-up result for certain solutions with high initial energy. However, as far as we know, there is little information on the long-time behavior of global solutions for above problems. Whether the global solutions will exhibit a long-time dynamic behavior at low initial energy? Do both problems (1)–(3) have the global solutions and asymptotic property at high initial energy level? These questions are all opening, and we are now working on these problems. On the other hand, we note that the fractional partial differential equations have been applied in various areas of science, and their related theoretical results and applications have been investigated by some authors (see [31–33] and the references therein). The study of their qualitative properties is one of the hot topics. Do the conclusions of present paper also hold for the initial and boundary value problems of the fractional nonlinear metaparabolic equations? This question is very interesting and opening.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

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