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Research Article

Lepton Mixing Patterns from $PSL_2(7)$ with a Generalized CP Symmetry

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Lepton mixing patterns from the modular group $PSL_2(7)$ with generalized CP symmetries are studied. The residual symmetries in both charged lepton and neutrino sectors are $Z_2 \times CP$. Seven types of mixing patterns at the 3σ level of the new global fit data are obtained. Among these patterns, three types of patterns can give the Dirac CP phase which is in the 1σ range of the global fit data. The effective mass of neutrinoless double-beta decay for these patterns is also examined.

1. Introduction

CP violation in the hadron sector was observed in 1964 [1]. Whether there is a counterpart in the lepton sector is still a mystery. Recent neutrino oscillation experiments show that the 1-3 mixing angle of leptons is nonzero [2–5]. It intrigues experiments to detect the Dirac CP-violating phase. Particularly, some fit results [6, 7] hint that this phase is around $-\pi/2$. In the theoretical respect, how to predict nontrivial lepton CP phases is interesting. In order to obtain mixing parameters of leptons, discrete flavor symmetries are widely used [8–41]. However, if no perturbation is considered, only finite groups of large orders could accommodate the results of new experiments [35]. Furthermore, they give a trivial Dirac CP-violating phase [35]. In order to improve predictions of flavor groups, some efforts have been made in generalizations of symmetries [42, 43]. Particularly, an intriguing method called generalized CP (GCP) symmetry was introduced [44–66]. In this scenario, the leptonic lagrangian satisfies both flavor and GCP symmetries. After spontaneous symmetries breaking, the residual flavor and GCP symmetries constrain the structures of mass matrices of leptons. Then, information on leptonic mixing angles and CP phases is obtained. From groups S_4 and A_5 with GCP symmetries, a trivial or maximal Dirac CP phase is obtained [46, 56]. The maximal Dirac phase satisfies the 1σ constraint from the new recent global fit data in case of inverted mass ordering [67]. However, it is not in the 1σ range for the normal mass ordering. S_4 and A_5 are small modular groups. We want to know whether a large one could give a more fit CP phase.

In this paper, we study the predictions of the modular group $PSL_2(7)$ with the GCP symmetry in the case of Majorana neutrinos. We suppose that residual symmetries in neutrino and charged lepton sectors are both $Z_2 \times CP$. Here, CP denotes a GCP symmetry. After examinations of combinations of residual symmetries, we find seven types of mixing patterns at the 3σ level of the fit data [67]. Among them, three types satisfy the 1σ constraint. So the group $PSL_2(7)$ with the GCP symmetry may serve as a candidate for explanations to experiment accommodable mixing patterns. We note that lepton mixing patterns from large finite modular group have been studied in the recent Refs. [32, 68, 69]. In Refs. [32, 68], no GCP symmetry is considered. In Ref. [32], residual GCP symmetries are considered either in the neutrino sector or the charged lepton sector. Namely, there is only one unfixed parameter in the lepton mixing matrix. Here, we consider the case that residual GCP symmetries constrain both charged leptons and neutrinos. So two parameters are contained in our mixing patterns.

This paper is organised as follows. In Section 2, the framework for the application of the group $PSL_2(7)$ with the GCP symmetry is introduced. In Section 3, the results from examination of the residual symmetries are presented. Finally, a summary is made.

2. Framework

In this section, we describe the basic facts of the group $PSL_2(7)$ and introduce the method of deriving lepton mixing patterns from the residual flavor and GCP symmetries.

2.1. Group Theory of $PSL_2(7)$

2.1.1. Generic Facts. The group $PSL_2(7)$ is also named $\sum(168)$. It could be constructed with two generators which satisfy following relations [68]:

$$\begin{aligned} S^2 &= T^7 = E, \\ (ST)^3 &= \left(ST^{-1}ST\right)^4 = E, \end{aligned} \quad (1)$$

where E is the identity element. This group has 6 conjugacy classes listed as follows [68]:

$$\begin{aligned} 1C_1 &: E, \\ 21C_2 &: S, \\ 56C_3 &: ST, \\ 42C_4 &: ST^3, \\ 24C_7^1 &: T, \\ 24C_7^2 &: T^3, \end{aligned} \quad (2)$$

where iC_j denotes that the class contains i elements of order j. Accordingly, there are 6 irreducible representations, namely,

$$\mathbf{1}, \mathbf{3}, \mathbf{3}^*, \mathbf{6}, \mathbf{7}, \mathbf{8}. \quad (3)$$

Without loss of generality, we consider the 3-dimensional representation **3** in the following sections. Accordingly, the generators could be expressed as [68]

$$\begin{aligned} S &= \frac{2}{\sqrt{7}} \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ s_3 & s_1 & -s_2 \end{pmatrix}, \\ T &= \begin{pmatrix} \varphi_7^2 & 0 & 0 \\ 0 & \varphi_7 & 0 \\ 0 & 0 & \varphi_7^{*3} \end{pmatrix}, \end{aligned} \quad (4)$$

where $s_k = \sin(k\pi/7)$, $\varphi_7 = e^{i2\pi/7}$.

Resorting to the conjugacy classes, we can obtain abelian subgroups of $PSL_2(7)$. These groups are candidates of the residual symmetries for leptons. In this paper, we consider the residual symmetry $Z_2 \times CP$ for leptons. So Z_2 subgroups are relevant. There are 21 Z_2 subgroups which are identified with the generators of them [68], i.e.,

$$\begin{aligned} A_1 &: S, \\ A_2 &: T^2ST^3ST, \\ A_3 &: TST^3ST^2, \\ A_4 &: T^4ST^3, \\ A_5 &: T^3ST^4, \\ A_6 &: T^2ST^4ST^2, \\ A_7 &: ST^2ST^4ST^2S, \\ A_8 &: ST^4ST^3S, \\ A_9 &: ST^3ST^4S, \\ A_{10} &: T^5ST^2, \\ A_{11} &: T^2ST^5, \\ A_{12} &: T^6ST, \\ A_{13} &: TST^6, \\ A_{14} &: ST^4ST^4, \\ A_{15} &: ST^3ST^3, \\ A_{16} &: ST^2ST, \\ A_{17} &: ST^5ST^6, \\ A_{18} &: \left(T^2ST^3S\right)^2, \\ A_{19} &: \left(T^5ST^4S\right)^2, \\ A_{20} &: \left(ST^3ST^4\right)^2, \\ A_{21} &: \left(ST^4ST^3\right)^2. \end{aligned} \quad (5)$$

2.1.2. Automorphism of $PSL_2(7)$. An automorphism of a group is a transformation which permutate elements of the group. These transformations form a group, namely, the automorphism group. For the group $PSL_2(7)$, the structure of the automorphism group is simple. It is listed as follows:

$$\begin{aligned} Z(PSL_2(7)) &= Z_1, \\ \mathrm{Aut}(PSL_2(7)) &\cong PSL(2, Z_7) \rtimes Z_2, \\ \mathrm{Inn}(PSL_2(7)) &\cong PSL_2(7), \\ \mathrm{Out}(PSL_2(7)) &\cong Z_2 = \{id, u\}, \end{aligned} \quad (6)$$

where Z, Aut, Inn, and Out denote the centre, the automorphism group, the inner automorphism, and the outer automorphism group, respectively. In detail, the inner automorphism group is composed of permutations of elements in the same conjugacy class. The outer automorphism group swaps conjugacy classes and representations. So it reflects the symmetries of the character table shown

Table 1: Character table of the group $PSL_2(7)$ [68].

Rep.	$1C_1$	$21C_2$	$56C_3$	$42C_4$	$24C_7^1$	$24C_7^2$
1	1	1	1	1	1	1
3	3	-1	0	1	$\varphi_7^* + \varphi_7^{*2} + \varphi_7^{*4}$	$\varphi_7 + \varphi_7^2 + \varphi_7^4$
3^*	3	-1	0	1	$\varphi_7 + \varphi_7^2 + \varphi_7^4$	$\varphi_7^* + \varphi_7^{*2} + \varphi_7^{*4}$
6	6	2	0	0	-1	-1
7	7	-1	1	-1	0	0
8	8	0	-1	0	1	1

in Table 1. The unique nontrivial outer automorphism of the group $PSL_2(7)$ is

$$u : 24C_7^1 \longleftrightarrow 24C_7^2, \quad \mathbf{3} \longleftrightarrow \mathbf{3}^*. \tag{7}$$

The representation of u could be obtained from its action on the generators S, T, i.e.,

$$u : S \longleftrightarrow S, \quad T \longleftrightarrow T^* = T^6. \tag{8}$$

In the 3-dimensional representation, the specific equations of the transformation read

$$X(u)S^* X^{-1}(u) = S^{-1} = S, \quad X(u)T^* X^{-1}(u) = T^{-1} = T^*. \tag{9}$$

The solution is

$$X(u) = e^{i\alpha} \operatorname{diag}(1,1,1). \tag{10}$$

Since the global phase is trivial for the lepton mixing patterns, we choose $e^{i\alpha} = 1$ in the following sections. A general automorphism is the product of the inner and the outer one. It could be expressed as

$$X(g_i) = \rho_3(g_i)X(u) = \rho_3(g_i), \text{with } g_i \in PSL_2(7), \tag{11}$$

where $\rho_3(g_i)$ is the 3-dimensional representation of the group element.

2.2. Approach

2.2.1. GCP Compatible with $PSL_2(7)$. The GCP transformation acts on the flavor space as

$$\Phi \longrightarrow X\Phi^C, \tag{12}$$

where Φ is a multiplet of fields, X is a unitary matrix, and Φ^C is the CP conjugation of Φ. In contrast, the flavor group acts on the fields as

$$\Phi \longrightarrow \rho(g_i)\Phi, \text{with } g_i \in PSL_2(7). \tag{13}$$

Accordingly, the consistence condition of GCP is [45]

$$\left(X^{-1}\rho(g)X\right)^* = \rho\left(g'\right). \tag{14}$$

Therefore, X is an automorphism of the flavor group $PSL_2(7)$. These GCP transformations form an automorphism group CP. The general theory satisfies the symmetry $PSL_2(7)\rtimes CP$. After fermions obtain masses from the vacuum expectation values of scalar fields, the original symmetry $PSL_2(7)\rtimes CP$ is broken to $G_e\rtimes CP_e$ in the charged lepton sector and $G_\nu\rtimes CP_\nu$ in the neutrino sector. Thus, the mass matrices of charged leptons and Majorana neutrinos satisfy the relations

$$\rho^+(g_e)m_e m_e^+\rho(g_e) = m_e m_e^+, \text{with } g_e \in G_e, \quad \rho^T(g_\nu)m_\nu\rho(g_\nu) = m_\nu, \text{with } g_\nu \in G_\nu. \tag{15}$$

The CP transformation X follows the relations

$$X_e^+ m_e m_e^+ X_e = (m_e m_e^+)^*, \quad \left(X_e^{-1}\rho(g_e)X_e\right)^* = \rho\left(g_e'\right), \quad X_\nu^T m_\nu X_\nu = m_\nu^*, \quad \left(X_\nu^{-1}\rho(g_\nu)X_\nu\right)^* = \rho\left(g_\nu'\right). \tag{16}$$

Since masses of leptons are nondegenerate, the CP transformation X should be a symmetric unitary matrix [56], i.e.,

$$X_\alpha = X_\alpha^T, \quad X_\alpha X_\alpha^* = E, \text{with } \alpha = e, \nu (\text{no sum}). \tag{17}$$

So X_α can be decomposed as $X_\alpha = \Omega_\alpha\Omega_\alpha^T$. This kind of CP transformations is called Bickerstaff-Damhus automorphism (BDA) [70, 71]. For the group $PSL_2(7)$, all BDAs in the 3-dimensional representation are listed as follows:

$$T^3ST^3, TST^4ST, TST^5ST, ST^3ST^3S, STST^4STS, STST^5STS, \quad E, T^i, ST^iS, \text{with } i = 1,2,3,4,5,6, \quad \left(T^2ST^2\right)^j, \left(ST^2ST^2S\right)^j, \left(T^2ST^5ST^2\right)^j, \text{with } j = 1,2,3. \tag{18}$$

2.2.2. Mixing Patterns from Residual Symmetries $Z_2 \times CP$. Once the residual symmetries are fixed, the lepton mixing pattern could be obtained up to permutations of rows and columns. In the direct method, the mixing matrix is completely determined by the symmetries. In the semidirect method, only several elements of the matrix are certain because of degeneracy of the eigenvalues of the residual symmetries. We concern on the semidirect method in this paper. The residual symmetry is $Z_{2e} \times CP_e, Z_{2\nu} \times CP_\nu$ in charged lepton and neutrino sectors, respectively. The consistence equation is written as

$$X\rho^*\left(g_{e,\nu}\right)X^* = \rho\left(g_{e,\nu}\right), \text{with } g_{e,\nu} \in Z_2. \tag{19}$$

Accordingly, the lepton mixing matrix $U_{\mathrm{PMNS}} \equiv U_e^{+} U_\nu$ is obtained from the matrix [46]

$$U_{e,\nu} = \Omega_{e,\nu} O(\theta_{e,\nu}) P_{e,\nu}, \tag{20}$$

where $R_{e,\nu}$ is a rotation matrix with an angle parameter $\theta_{e,\nu}$, and $P_{e,\nu}$ is a phase matrix, i.e.,

$$P_{e,\nu} = \mathrm{diag}\left(1, i^j, i^k\right), \text{with } j, k = 0, 1, 2, 3. \tag{21}$$

Because P_e gives nonphysical phases, it is omitted in the following sections.

2.2.3. Similar Transformations. In order to obtain viable mixing patterns, all possible combinations of residual symmetries should be examined. However, if two combinations are connected by a similarity transformation, namely,

$$\begin{aligned} \rho\left(g'_{e,\nu}\right) &= V\rho\left(g_{e,\nu}\right)V^{+}, \\ X'_{e,\nu} &= VX_{e,\nu}V^{T}, \end{aligned} \tag{22}$$

they would correspond to the same mixing matrix. Therefore, we could just examine nonequivalent combinations. In the following sections, $Z_{2\nu}$ is fixed on the subgroup Z_2^S which is generated by the group element S. The consistent GCP transformations for Z_2^S are listed as follows:

$$\begin{aligned} X_1 &= E, \\ X_2 &= S, \\ X_3 &= T^2ST^5ST^2, \\ X_4 &= \left(T^5ST^2ST^5\right) = \left(T^2ST^5ST^2\right)^{*}, \end{aligned} \tag{23}$$

where X_1 and X_2 correspond to the equivalent mixing patterns, so do X_3 and X_4. Z_{2e}, X_e can be obtained from the similar transformations. In detail, for generators of Z_2 subgroups, we have $\rho(A_i) = V_i S V_i^{+}$ with V_i listed as follows:

$$V_2 = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7^3 & 0 & 0 \\ 0 & -\varphi_7 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7^{*3} & 0 & 0 \\ 0 & -\varphi_7^{*} & 0 \end{pmatrix},$$

$$V_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7^3 & 0 \\ 0 & 0 & -\varphi_7 \end{pmatrix},$$

$$V_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7^{*3} & 0 \\ 0 & 0 & -\varphi_7^{*} \end{pmatrix},$$

$$V_6 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$V_7 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$V_8 = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7^{*} & 0 & 0 \\ 0 & -\varphi_7^2 & 0 \end{pmatrix},$$

$$V_9 = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7 & 0 & 0 \\ 0 & -\varphi_7^{*2} & 0 \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7^2 & 0 \\ 0 & 0 & \varphi_7^3 \end{pmatrix},$$

$$V_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7^{*2} & 0 \\ 0 & 0 & \varphi_7^{*3} \end{pmatrix},$$

$$V_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7 & 0 \\ 0 & 0 & \varphi_7^{*2} \end{pmatrix},$$

$$v_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_7^{*} & 0 \\ 0 & 0 & \varphi_7^2 \end{pmatrix},$$

$$v_{14} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7^3 \\ \varphi_7 & 0 & 0 \end{pmatrix},$$

$$v_{15} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7^{*3} \\ \varphi_7^{*} & 0 & 0 \end{pmatrix},$$

$$v_{16} = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7^2 & 0 & 0 \\ 0 & -\varphi_7^3 & 0 \end{pmatrix},$$

$$v_{17} = \begin{pmatrix} 0 & 0 & 1 \\ \varphi_7^{*2} & 0 & 0 \\ 0 & -\varphi_7^{*3} & 0 \end{pmatrix},$$

$$V_{18} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7^{*} \\ \varphi_7^{2} & 0 & 0 \end{pmatrix},$$

$$V_{19} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7 \\ \varphi_7^{*2} & 0 & 0 \end{pmatrix},$$

$$V_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7^{2} \\ \varphi_7^{3} & 0 & 0 \end{pmatrix},$$

$$V_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\varphi_7^{*2} \\ \varphi_7^{*3} & 0 & 0 \end{pmatrix}. \quad (24)$$

Particularly, V_1 is the identity matrix. So a general combination of the residual symmetries is of the form

$$\left(Z_{2e}^{Ai}, X_{ej}(A_i), Z_{2\nu}^{S}, X_{\nu k}(S)\right), \quad (25)$$

with j, $k = 1, 3$. The corresponding lepton mixing matrix is written as

$$U_{\text{PMNS}} = O^T(\theta_e)\Omega_j^{+} V_i^{+} \Omega_k O(\theta_\nu) P_\nu, \quad (26)$$

where

$$O(\theta_{e,\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{e,\nu} & \sin\theta_{e,\nu} \\ 0 & -\sin\theta_{e,\nu} & \cos\theta_{e,\nu} \end{pmatrix},$$
$$\Omega_1 = \begin{pmatrix} r_1 & -\sin\theta_1 & -\cos\theta_1 \sin\theta_2 \\ r_2 & 0 & \cos\theta_2 \\ r_3 & \cos\theta_1 & -\sin\theta_1 \sin\theta_2 \end{pmatrix}, \quad (27)$$
$$\Omega_3 = \Omega_1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4}\cos\phi & e^{i3\pi/4}\sin\phi \\ 0 & -e^{i\pi/4}\sin\phi & e^{i3\pi/4}\cos\phi \end{pmatrix},$$

with

$$\begin{aligned} r_1 &= 2\sqrt{\frac{2}{7}} \sin\frac{2\pi}{7} \sin\frac{3\pi}{7}, \\ r_2 &= 2\sqrt{\frac{2}{7}} \sin\frac{\pi}{7} \sin\frac{2\pi}{7}, \\ r_3 &= 2\sqrt{\frac{2}{7}} \sin\frac{\pi}{7} \sin\frac{3\pi}{7}, \\ \theta_1 &= \arcsin\frac{r_3}{\sqrt{r_1^2 + r_3^2}}, \\ \theta_2 &= \arcsin\frac{r_2\cos\theta_1}{\sqrt{r_1^2 + r_2^2\cos^2\theta_1}}, \\ \phi &= \arcsin\frac{1}{\sqrt{1 + x_1^2}}, \end{aligned} \quad (28)$$

where x_1 is a real root of the equation

$$x^{12} - 48x^{10} + 323x^8 - 608x^6 + 323x^4 - 48x^2 + 1 = 0, \quad (29)$$

$x_1 \simeq \pm 0.449807$.

3. Results

3.1. Viable Mixing Matrices from Combinations of Residual Symmetries. Given the recent global fit data of neutrino oscillations [67], we examine the predictions of combinations of residual symmetries of the form $(Z_{2e}^{Ai}, X_{ej}(A_i), Z_{2\nu}^{S}, X_{\nu k}(S))$ with the χ^2 function defined as

$$\chi^2 = \sum_{ij=13,23,12} \left(\frac{\sin^2\theta_{ij} - \left(\sin^2\theta_{ij}\right)^{ex}}{\sigma_{ij}} \right)^2, \quad (30)$$

where $(\sin^2\theta_{ij})^{ex}$ is the best fit data from Ref. [67] and σ_{ij} is the 1σ error. The viable combinations at the 3σ level (up to equivalent ones) are listed as follows.

Type Ia:

$$\left(Z_{2e}^{ST^4ST^4}, X_{3e} = V_{16}\left(T^2ST^5ST^2\right)V_{16}^T, Z_{2\nu}^S, X_{1\nu} = E\right), \quad (31)$$

Type Ib:

$$\left(Z_{2e}^{ST^4ST^4}, X_{1e} = V_{16}V_{16}^T, Z_{2\nu}^S, X_{3\nu} = T^2ST^5ST^2\right), \quad (32)$$

Type Ib*:

$$\left(Z_{2e}^{ST^3ST^3}, X_{1e} = V_{17}V_{17}^T, Z_{2\nu}^S, X_{3\nu} = T^2ST^5ST^2\right), \quad (33)$$

Type IIa:

$$\left(Z_{2e}^{T^4ST^3}, X_{1e} = V_4V_4^T, Z_{2\nu}^S, X_{1\nu} = E\right), \quad (34)$$

Type IIa*:

$$\left(Z_{2e}^{T^3ST^4}, X_{1e} = V_5V_5^T, Z_{2\nu}^S, X_{1\nu} = E\right), \tag{35}$$

Type IIb:

$$\left(Z_{2e}^{T^4ST^3}, X_{1e} = V_4V_4^T, Z_{2\nu}^S, X_{3\nu} = T^2ST^5ST^2\right), \tag{36}$$

Type IIb*:

$$\left(Z_{2e}^{T^3ST^4}, X_{1e} = V_5V_5^T, Z_{2\nu}^S, X_{3\nu} = T^2ST^5ST^2\right). \tag{37}$$

The corresponding mixing matrices are dependent on permutations of rows and columns. For every combination, the matrix which fits the data best is listed as follows.

$$\begin{aligned}
U_{\mathrm{Ia}} &= S_{13}O^T(\theta_e)\Omega_3^+V_{16}^+\Omega_1 O(\theta_\nu)P_\nu, \\
U_{\mathrm{Ib}} &= S_{12}O^T(\theta_e)\Omega_1^+V_{16}^+\Omega_3 O(\theta_\nu)P_\nu S_{12}, \\
U_{\mathrm{Ib}^*} &= U_{\mathrm{Ib}}^* = S_{12}O^T(\theta_e)\Omega_1^+V_{17}^+\Omega_3^* O(\theta_\nu)P_\nu S_{12}, \\
U_{\mathrm{IIa}} &= S_{13}O^T(\theta_e)\Omega_1^+V_4^+\Omega_1 O(\theta_\nu)P_\nu, \\
U_{\mathrm{IIa}^*} &= U_{\mathrm{IIa}}^* = S_{13}O^T(\theta_e)\Omega_1^+V_5^+\Omega_1 O(\theta_\nu)P_\nu, \\
U_{\mathrm{IIb}} &= S_{12}O^T(\theta_e)\Omega_1^+V_4^+\Omega_3 O(\theta_\nu)P_\nu, \\
U_{\mathrm{IIb}^*} &= U_{\mathrm{IIb}}^* = S_{12}O^T(\theta_e)\Omega_1^+V_5^+\Omega_3^* O(\theta_\nu)P_\nu,
\end{aligned} \tag{38}$$

where

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{39}$$

We note that the mixing matrices except U_{Ia} are paired through the complex conjugation. The predictions of the matrices in a pair are identical except the signs of the CP phases. So we can consider U_{Ia}, U_{Ib}, U_{IIa}, and U_{IIb} as representatives.

3.2. Mixing Angles and CP Invariants. Employing the parametrization of the form

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha/2} & 0 \\ 0 & 0 & e^{i(\beta/2+\delta)} \end{pmatrix}, \tag{40}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, δ is the Dirac CP-violating phase, and α and β are Majorana phases; we could obtain lepton mixing angles and the CP invariants J_{cp} [72], J_1, and J_2 defined as

$$\begin{aligned}
J_{\mathrm{cp}} &\equiv \mathrm{Im}\,[U_{11}U_{13}^*U_{31}^*U_{33}] = \frac{1}{8}\sin 2\theta_{13}\sin 2\theta_{23}\sin 2\theta_{12}\cos\theta_{13}\sin\delta, \\
J_1 &\equiv \mathrm{Im}\left[(U_{11}^*)^2U_{12}^2\right] = \sin^2\theta_{12}\cos^2\theta_{12}\cos^4\theta_{13}\sin\alpha, \\
J_2 &\equiv \mathrm{Im}\left[(U_{11}^*)^2U_{13}^2\right] = \sin^2\theta_{13}\cos^2\theta_{13}\cos^2\theta_{12}\sin\beta.
\end{aligned} \tag{41}$$

Their specific forms are listed as follows.

U_{Ia}:

$$\begin{aligned}
\sin^2\theta_{13}(\theta_e,\theta_\nu) &\simeq 0.3125 - 0.1758\cos 2\theta_\nu + 0.02073\cos(2\theta_\nu - 2\theta_e) - 0.1654\cos 2\theta_e + 0.2786\cos(2\theta_\nu + 2\theta_e) + 0.065086\sin 2\theta_\nu - 0.02338\sin(2\theta_\nu - 2\theta_e) - 0.08839\sin 2\theta_e + 0.03824\sin(2\theta_\nu - 2\theta_e), \\
\sin^2\theta_{23}(\theta_e,\theta_\nu) &\simeq \frac{\left[0.002647\cos^2\theta_e\cos^2\theta_\nu + 0.05023\cos^2\theta_e\sin 2\theta_\nu + 0.9531\cos^2\theta_e\sin^2\theta_\nu + 0.02677\sin 2\theta_e\cos^2\theta_\nu + 0.2579\sin 2\theta_e\sin 2\theta_\nu + 0.27067\sin^2\theta_e\cos^2\theta_\nu + 0.150\sin 2\theta_e\sin^2\theta_\nu + 0.07994\sin^2\theta_e\sin 2\theta_\nu + 0.02361\sin^2\theta_e\sin^2\theta_\nu\right]}{1 - \sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
\sin^2\theta_{12}(\theta_e,\theta_\nu) &= \frac{\sin^2\theta_{23}(\theta_e + \pi/2, \theta_\nu + \pi/2)\left(1 - \sin^2\theta_{13}(\theta_e + \pi/2, \theta_\nu + \pi/2)\right)}{1 - \sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
J_{\mathrm{cp}}(\theta_e,\theta_\nu) &= 0, \\
J_1(\theta_e,\theta_\nu) &= 0, \\
J_2(\theta_e,\theta_\nu) &= 0.
\end{aligned} \tag{42}$$

U_{Ib}:

$$
\begin{aligned}
\sin^2\theta_{13}(\theta_e,\theta_\nu) \simeq\ & 0.2438\cos^2\theta_e\cos^2\theta_\nu + 0.1871\cos^2\theta_e\sin 2\theta_\nu \\
& + 0.01618\cos^2\theta_e\sin^2\theta_\nu + 0.264\sin 2\theta_e\cos^2\theta_\nu \\
& + 0.1659\sin 2\theta_e\sin 2\theta_\nu - 0.16566\sin 2\theta_e\sin^2\theta_\nu \\
& + 0.712\sin^2\theta_e\cos^2\theta_\nu - 0.187\sin^2\theta_e\sin 2\theta_\nu \\
& + 0.13245\sin^2\theta_e\sin^2\theta_\nu, \\
\sin^2\theta_{23}(\theta_e,\theta_\nu) \simeq\ & \frac{[0.375 - 0.3307\cos 2\theta_\nu]}{1-\sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
\sin^2\theta_{12}(\theta_e,\theta_\nu) \simeq\ & \frac{[0.375 + 0.2194\cos 2\theta_e - 0.2475\sin 2\theta_e]}{1-\sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
J_{\mathrm{cp}}(\theta_e,\theta_\nu) \simeq\ & 0.001543 + 0.008446\cos 2\theta_\nu \\
& + 0.006903\cos 4\theta_\nu \\
& + 0.002733\cos(2\theta_\nu - 4\theta_e) \\
& - 0.003911\cos(4\theta_\nu - 2\theta_e) \\
& + 0.003405\cos(2\theta_\nu - 2\theta_e) \\
& + 0.000384\cos(4\theta_\nu - 4\theta_e) \\
& - 0.002896\cos 2\theta_e \\
& + 0.001353\cos 4\theta_e \\
& - 0.008932\cos(2\theta_\nu + 2\theta_e) \\
& - 0.002234\cos(4\theta_\nu + 4\theta_e) \\
& - 0.00114\cos(4\theta_\nu + 2\theta_e) \\
& - 0.00323\cos(2\theta_\nu + 4\theta_e) \\
& - 0.00951\sin 2\theta_\nu \\
& + 0.000585\sin 4\theta_\nu \\
& + 0.002935\sin(2\theta_\nu - 4\theta_e) \\
& - 0.000292\sin(4\theta_\nu - 2\theta_e) \\
& - 0.02566\sin(2\theta_\nu - 2\theta_e) \\
& + 0.000304\sin(4\theta_\nu - 4\theta_e) \\
& + 0.00328\sin 2\theta_e \\
& - 0.00591\sin 4\theta_e \\
& - 0.000519\sin(2\theta_\nu + 2\theta_e) \\
& + 0.00715\sin(4\theta_\nu + 4\theta_e) \\
& - 0.00775\sin(4\theta_\nu + 2\theta_e) \\
& + 0.00388\sin(2\theta_\nu + 4\theta_e), \\
J_1(\theta_e,\theta_\nu) \simeq\ & \pm[0.05167 - 0.00391\cos 2\theta_\nu \\
& - 0.00304\cos(2\theta_\nu - 4\theta_e) \\
& - 0.01371\cos(2\theta_\nu - 2\theta_e) \\
& + 0.01037\cos 2\theta_e \\
& + 0.006185\cos 4\theta_e \\
& + 0.04114\cos(2\theta_\nu + 2\theta_e) \\
& - 0.00491\cos(2\theta_\nu + 4\theta_e) \\
& + 0.02521\sin(2\theta_\nu - 4\theta_e) \\
& - 0.01546\sin(2\theta_\nu - 2\theta_e) \\
& - 0.1169\sin 2\theta_e \\
& + 0.0153\sin 4\theta_e \\
& - 0.0464\sin(2\theta_\nu + 2\theta_e) \\
& - 0.0407\sin(2\theta_\nu + 4\theta_e)], \\
J_2(\theta_e,\theta_\nu) \simeq\ & \pm[0.00726\cos(2\theta_\nu - 4\theta_e) \\
& - 0.0248\cos(2\theta_\nu - 2\theta_e) \\
& + 0.05787\cos(2\theta_\nu + 2\theta_e) \\
& - 0.05079\cos(2\theta_\nu + 2\theta_e) \\
& - 0.07308\sin 2\theta_\nu \\
& - 0.000875\sin(2\theta_\nu - 4\theta_e) \\
& + 0.02199\sin(2\theta_\nu - 2\theta_e) \\
& + 0.0513\sin(2\theta_\nu + 2\theta_e) \\
& + 0.00612\sin(2\theta_\nu + 4\theta_e)].
\end{aligned}
\tag{43}
$$

U_{IIa}:

$$
\begin{aligned}
\sin^2\theta_{13}(\theta_e,\theta_\nu) \simeq\ & 0.616\cos^2\theta_e\cos^2\theta_\nu - 0.133\cos^2\theta_e\sin 2\theta_\nu + 0.0769\cos^2\theta_e\sin^2\theta_\nu - 0.1328\sin 2\theta_e\cos^2\theta_\nu + 0.01762\sin 2\theta_e\sin 2\theta_\nu - 0.1322\sin 2\theta_e\sin^2\theta_\nu \\
& + 0.0769\sin^2\theta_e\cos^2\theta_\nu - 0.1322\sin^2\theta_e\sin 2\theta_\nu + 0.4155\sin^2\theta_e\sin^2\theta_\nu, \\
\sin^2\theta_{23}(\theta_e,\theta_\nu) \simeq\ & \frac{[0.2963 + 0.0502\cos 2\theta_\nu - 0.1185\cos(2\theta_\nu - 2\theta_e) - 0.0502\cos 2\theta_e - 0.101\cos(2\theta_\nu + 2\theta_e) - 0.01325\sin 2\theta_\nu + 0.01325\sin 2\theta_e + 0.000294\sin(2\theta_\nu + 2\theta_e)]}{1-\sin^2\theta_{13}}, \\
\sin^2\theta_{12}(\theta_e,\theta_\nu) =\ & \frac{\sin^2\theta_{23}(\theta_e+\pi/2,\theta_\nu+\pi/2)\left(1-\sin^2\theta_{13}(\theta_e+\pi/2,\theta_\nu+\pi/2)\right)}{1-\sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
J_{\mathrm{CP}}(\theta_e,\theta_\nu) \simeq\ & 0.00919\cos(2\theta_\nu - 2\theta_e) - 0.01601\cos(2\theta_\nu + 2\theta_e) + 0.06163\sin(2\theta_\nu + 2\theta_e), \\
J_1(\theta_e,\theta_\nu) \simeq\ & \pm[-0.00475 - 0.0474\cos 2\theta_\nu + 0.0144\cos(2\theta_\nu - 4\theta_e) \\
& + 0.0203\cos(2\theta_\nu - 2\theta_e) + 0.0541\cos 2\theta_e - 0.0461\cos 4\theta_e + 0.0188\cos(2\theta_\nu + 2\theta_e) \\
& - 0.03285\cos(2\theta_\nu + 4\theta_e) + 0.002649\sin 2\theta_\nu - 0.0101\sin(2\theta_\nu - 4\theta_e) + 0.02699\sin(2\theta_\nu - 2\theta_e) \\
& + 0.02349\sin 2\theta_e + 0.05424\sin 4\theta_e - 0.01937\sin(2\theta_\nu + 2\theta_e) - 0.04581\sin(2\theta_\nu + 4\theta_e)], \\
J_2(\theta_e,\theta_\nu) =\ & J_1\left(\theta_e,\theta_\nu + \frac{\pi}{2}\right).
\end{aligned}
\tag{44}
$$

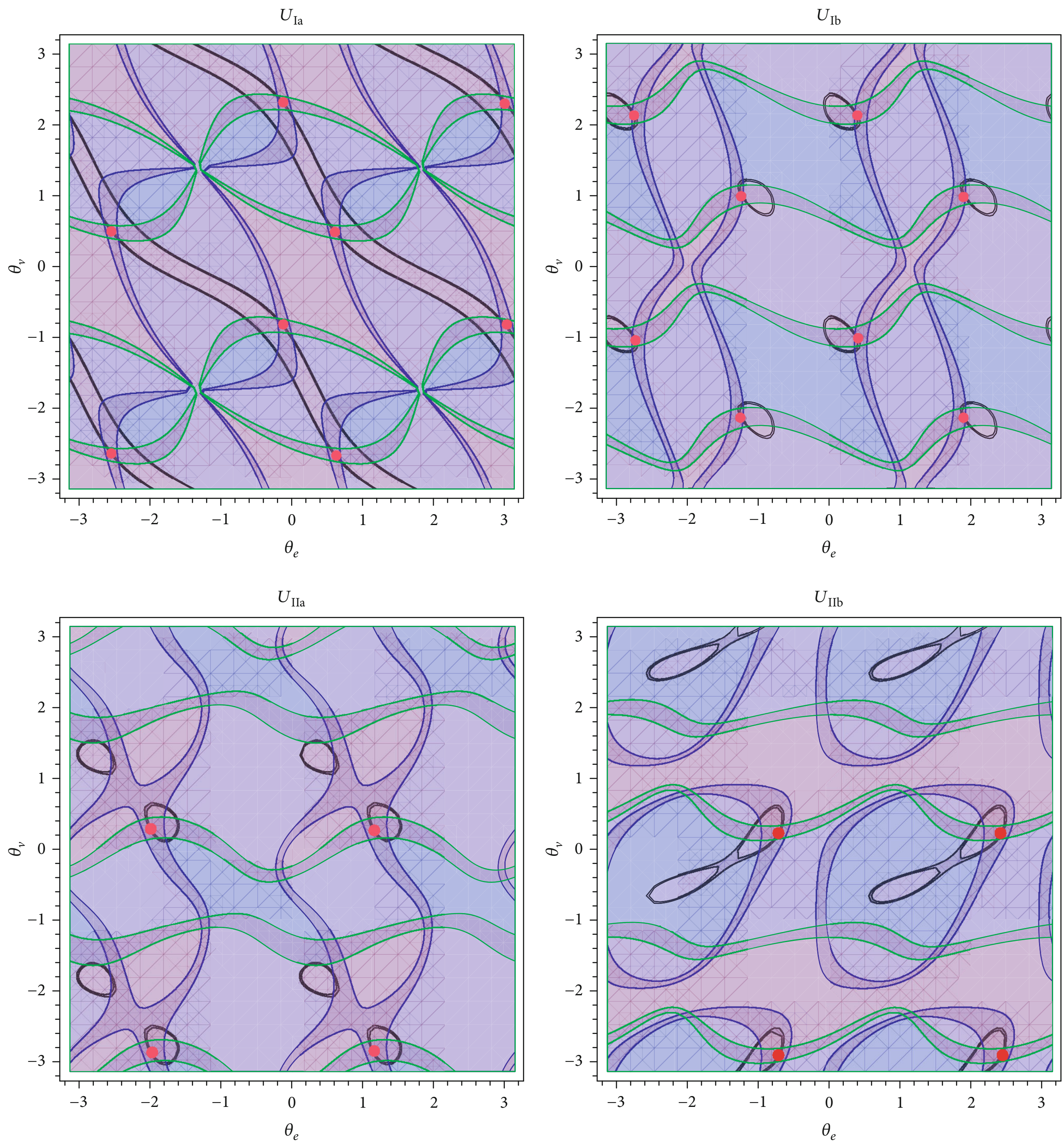

FIGURE 1: Parameter space for the mixing patterns constrained by the global fit data [67] at the 3σ level in the normal mass ordering (NO). The parameter spaces for θ_{12} are the strips with blue boundaries, and those for θ_{23} are with green boundaries. The strips for θ_{13} are tiny, i.e., almost reduced to black curves. Their intersection areas are signed by the red dots. The parameter spaces in the case of inverted mass ordering (IO) are similar. So they are not shown here.

Table 2: Best fit data of lepton mixing angles and CP phases. "N" and "I" denote the normal ordering of neutrino masses and the inverted ordering, respectively.

Patterns	$(\theta_e^{bf}, \theta_\nu^{bf})$	χ^2_{min}	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$	$\sin^2\theta_{12}$	$\lvert\sin\delta\rvert$	$\lvert\sin\alpha\rvert$	$\lvert\sin\beta\rvert$
Ia(N)	$(0.18725\pi, 0.1644\pi)$, $(-0.0297\pi, 0.72267\pi)$	0.000016	0.0216	0.547	0.320	0	0	0
Ia(I)	$(0.18596\pi, 0.16544\pi)$, $(-0.0297\pi, 0.72172\pi)$	0.021	0.0220	0.549	0.318	0	0	0
Ib(N)	$(-0.4028\pi, 0.3393\pi)$, $(-0.8663\pi, -0.3393\pi)$	2.33	0.0217	0.563	0.345	0.813	0.952	0.43
Ib(I)	$(-0.4037\pi, 0.340\pi)$, $(-0.8654\pi, -0.340\pi)$	1.97	0.0221	0.565	0.343	0.817	0.953	0.42
IIa(N)	$(0.3694\pi, 0.0767\pi)$	0.064	0.0216	0.549	0.316	0.997	0.769	0.588
IIa(I)	$(0.3701\pi, 0.0745\pi)$	0.057	0.022	0.553	0.316	0.998	0.774	0.547
IIb(N)	$(-0.2345\pi, 0.06297\pi)$	0.79	0.02156	0.5507	0.304	0.800	0.617	0.747
IIb(I)	$(-0.2348\pi, 0.06145\pi)$	0.81	0.0220	0.5546	0.304	0.742	0.616	0.730

U_{IIb}:

$$\begin{aligned}
\sin^2\theta_{13}(\theta_e,\theta_\nu) &\simeq 0.2963 - 0.1324\cos 2\theta_\nu \\
&\quad + 0.199\cos(2\theta_\nu - 2\theta_e) \\
&\quad - 0.0502\cos 2\theta_e - 0.05315\cos(2\theta_\nu + 2\theta_e) \\
&\quad + 0.1464\sin 2\theta_\nu - 0.00634\sin(2\theta_\nu - 2\theta_e) \\
&\quad + 0.1325\sin 2\theta_e - 0.01973\sin(2\theta_\nu + 2\theta_e), \\
\sin^2\theta_{23}(\theta_e,\theta_\nu) &\simeq \frac{[0.4074 + 0.2648\cos 2\theta_\nu - 0.2927\sin 2\theta]}{1-\sin^2\theta_{13}}, \\
\sin^2\theta_{12}(\theta_e,\theta_\nu) &= \frac{\sin^2\theta_{13}(\theta_e,\theta_\nu+\pi/2)}{1-\sin^2\theta_{13}(\theta_e,\theta_\nu)}, \\
J_{CP}(\theta_e,\theta_\nu) &\simeq 0.0261\cos(2\theta_\nu - 2\theta_e) \\
&\quad + 0.02447\cos(2\theta_\nu + 2\theta_e) \\
&\quad + 0.04\sin(2\theta_\nu - 2\theta_e) \\
&\quad + 0.01804\sin(2\theta_\nu + 2\theta_e), \\
J_1(\theta_e,\theta_\nu) &\simeq \pm[0.01355 + 0.01357\cos 2\theta_\nu \\
&\quad + 0.02594\cos(2\theta_\nu - 4\theta_e) \\
&\quad + 0.00115\cos(2\theta_\nu - 2\theta_e) \\
&\quad - 0.02127\cos 2\theta_e \\
&\quad + 0.0643\cos 4\theta_e \\
&\quad - 0.03256\cos(2\theta_\nu + 2\theta_e) \\
&\quad + 0.0213\cos(2\theta_\nu + 4\theta_e) - 0.03385\sin 2\theta_\nu \\
&\quad - 0.04857\sin(2\theta_\nu - 4\theta_e) \\
&\quad - 0.02714\sin(2\theta_\nu - 2\theta_e) - 0.001069\sin 2\theta_e \\
&\quad + 0.03351\sin 4\theta_e - 0.007202\sin(2\theta_\nu + 2\theta_e) \\
&\quad - 0.002316\sin(2\theta_\nu + 4\theta_e)], \\
J_2(\theta_e,\theta_\nu) &= J_1\left(\theta_e, \theta_\nu + \frac{\pi}{2}\right).
\end{aligned} \tag{45}$$

The sign of J_1 and J_2 is uncertain. It is dependent on the index j and k in the diagonal phase matrix P_ν.

3.3. Constraints on (θ_e, θ_ν) and Best Fit Data. The parameter space of (θ_e, θ_ν) is shown in Figure 1. The best fit data of lepton mixing angle and CP phases are listed in Table 2. We make some comments on the numerical results:

(i) In a period of the parameters (θ_e, θ_ν), there are two best fit points for the mixing matrices U_{Ia} and U_{Ib}. For U_{Ib}, these two points give the same magnitude of $\sin\delta$ with different signs. For the mixing matrices U_{IIa} and U_{IIb}, there is only one best fit point in a period. $\sin\delta$ for them are both positive. Even so, U^*_{IIa} and U^*_{IIb} could give negative $\sin\delta$ while the mixing angles are kept the same. (ii) The best fit value of θ_{23} from the global fit data [67] is in the second octant. Accordingly, our fit data is in the same octant. In the case of normal mass ordering (NO), the best fit value of δ in U_{Ib} and U^*_{IIb} could be around -0.3π. It is in the 1σ range of the global fit data. In the case of inverted mass ordering (IO), the best fit value of δ in U^*_{IIa} is around -0.5π. It is also in the 1σ range of the global fit data.

3.4. The Effective Mass of Neutrinoless Double-Beta Decay $<m_{ee}>$. Although the residual symmetries ($Z_{2e}\times CP_e$, $Z_{2\nu}\times CP_\nu$) cannot restrain the masses of neutrinos m_i with $i = 1,2,3$, they may affect the effective mass of neutrinoless double-beta decay $<m_{ee}>$ through the mixing angles and Majorana phases. Here, $<m_{ee}>$ is expressed as

$$\langle m_{ee}\rangle \equiv \left|m_1 U_{11}^2 + m_2 U_{12}^2 + m_3 U_{13}^2\right|. \tag{46}$$

Employing the lepton mixing matrix $U(\theta_e^{bf}, \theta_\nu^{bf})$ and the best fit data on Δm_{12}^2, $|\Delta m_{13}^2|$ [67], we plot $<m_{ee}>$ against the mass of the lightest neutrino m_0 in Figure 2. For (θ_e, θ_ν) taken from the 3σ range around the best fit data, the curves of $<m_{ee}>$ in every pattern are shown in Figure 3. We make some comments on the main results shown in these figures.

(i) In the case of IO, these patterns give stringent constraints on the ranges of $<m_{ee}>$. Particularly, $<m_{ee}>$ for patterns with the indexes (1, 0) and (1, 1) is independent of the parameters (θ_e, θ_ν).

In the range of m_0 favored by cosmology, $<m_{ee}>$ is around 0.045 eV for patterns Ia, IIa, and IIb. For pattern Ib, it is 0.04 eV. These values approximate the upper limit from the global fit data at the 3σ level. They are in the reach of future double-beta decay experiments [73]. Interestingly, similar observations still hold for the patterns from the group S_4 with GCP [62]

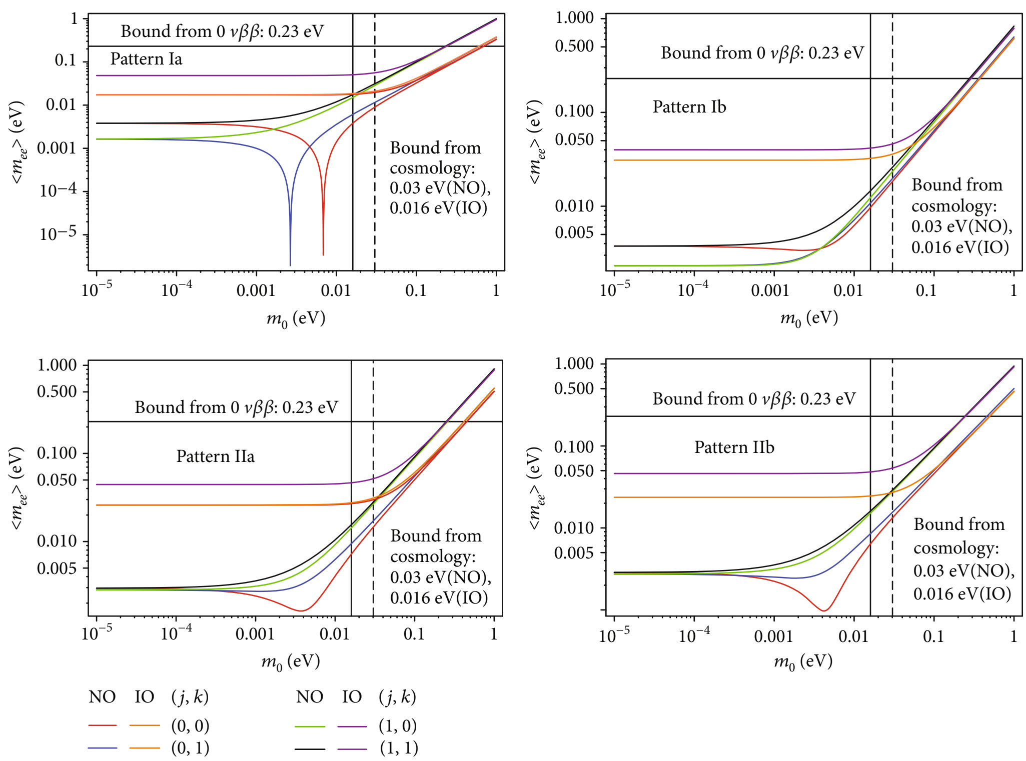

Figure 2: The effective mass of neutrinoless double-beta decay $\langle m_{ee} \rangle$ against the mass of the lightest neutrino m_0 with the best fit data $(\theta_e^{bf}, \theta_\nu^{bf})$. The bound on m_0 from cosmology is taken from Ref. [74]. The constraint on $\langle m_{ee} \rangle$ is from Ref. [75]. The best fit data of Δm_{12}^2 and $|\Delta m_{13}^2|$ is from Ref. [67]. The legends for every pattern are shown in the top left panel. The indexes (j, k) are defined in Equation (21). The best fit data $(\theta_e^{bf}, \theta_\nu^{bf})$ for pattern Ia and that for pattern Ib take the second one in Table 2, respectively.

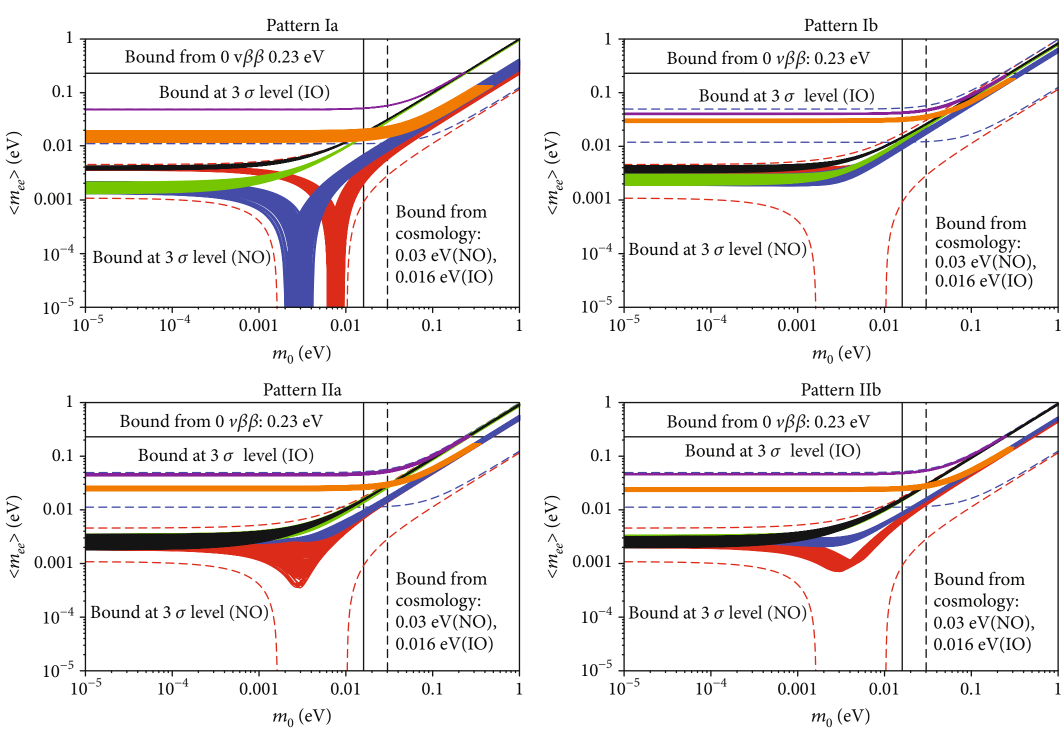

Figure 3: The effective mass of neutrinoless double-beta decay $\langle m_{ee} \rangle$ against the mass of the lightest neutrino m_0 in the 3σ ranges of (θ_e, θ_ν). The conventions follow those in Figure 2. The dashed boundary lines at the 3σ level are obtained from the global fit data [67].

(ii) In the case of NO, the variance of $<m_{ee}>$ is noticeable for every pattern. Particularly, for patterns Ia and Ib, $<m_{ee}>$ with the indexes (1, 1) could reach the upper limit at the 3σ level. Even so, it is not accessible to near future experiments

(iii) In both NO and IO case, without the precise constraint on the Dirac CP phase, these four mixing patterns cannot be discriminated by future double-beta decay experiments because of the overlaps of ranges of $<m_{ee}>$

4. Summary

For the group $PSL_2(7)$ with GCP symmetries, the predictions of the residual symmetries $Z_2 \times CP$ in both neutrino and charged lepton sectors are examined. Seven types of viable mixing patterns at the 3σ level of the global fit data are obtained. Among them, six types are paired through the complex conjugation. Three types of patterns can give the Dirac CP phase which is in the 1σ range of the global fit data. With the parameters (θ_e, θ_ν), the constraints of residual symmetries on the effective mass of neutrinoless double-beta decay are also examined. In the case of IO, every pattern can give the effective mass accessible to the future experiments.

Data Availability

The global fit data supporting this research paper are from previously reported studies, which have been cited. The processed data are freely available.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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