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Fixed Point Theorems for Mappings Satisfying Implicit Relation in Partial Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this manuscript, we shall introduce the implicit relation on ${\mathbb{R}_+}^4$. Using this implicit relation, we shall prove fixed point and common fixed point theorems on partial metric spaces.

Keywords: Partial metric space; implicit relation; common fixed point.

1 Introduction

In fixed point theorem, the widely known Banach contraction principle is the most popular result. This principle states that a contraction map on a complete metric space has an unique fixed point. Firstly Kannan [1] gave a new contractive condition in 1969. The generalization of the Banach contraction condition is done by Chatterjae [2] in 1972.

The following is a description of the well-known Banach contraction principle.

Theorem 1.1 [3] Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping satisfying the contractive condition*:*

$d(T(x), T(y)) \leq \alpha d(x, y)$

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for all $x, y \in X$, where $\alpha \in [0,1)$ is a constant. Then T has a unique fixed point in X and that point can be obtained as a limit of repeated iteration of the mapping at any point of X."

Matthews ([4, 5]) introduced a new space called partial metric space. The analogous of Banach contraction principle was proved by Matthews [5], which made the partial metric applicable in fixed point theory. By generalization of partial metric function named as weak partial metric function Heckmann [6] entrenched some outcomes.

Some generalizations are given by many authors of the outcomes of Matthews ([7-11, 12-15]).

The introduction of implicit relations in common fixed point theorems was established by V. Popa [16] in 1999. Further many authors extended common fixed point theorems using implicit relations ([17, 18, 19, 20, 21]).

Definition 1.2 [4] Let X be a nonempty set and let $p: X \times X \to \mathbb{R}^+$ be a function satisfying:

$$(pm1) x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y),$$

$(pm2) p(x,x) \le p(x,y)$,
$(pm3) p(x,y) = p(y,x)$,
$(pm4) p(x,y) \le p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X, p) is called partial metric space.

It is clear that if $p(x,y) = 0$, then from $(pm1)$ and $(pm2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Example 1.3 [22] Let $\tilde{X} = \mathbb{R}^+$ and $p: \tilde{X} \times \tilde{X} \to \mathbb{R}^+$ given by $p(x,y) = max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 1.4 [22] Let $\tilde{X} = \{[\tilde{a}, \tilde{b}]: \tilde{a}, \tilde{b} \in \mathbb{R}, \tilde{a} \le \tilde{b}\}$. Then $p([\tilde{a}, \tilde{b}], [\tilde{c}, \tilde{d}] = max\{\tilde{b}, \tilde{d}\} - min\{\tilde{a}, \tilde{c}\}$ defines a partial metric p on $\tilde{X}$.

Lemma 1.5 ([4, 5])Let (X, p) be a partial metric space. Then:

$(c1)$ a sequence x_n in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_p), $(c2)(X, p)$ is complete if and only if the metric space (X, d_p) is complete, $(c3)$ a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 1.6 [8]Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z,z) = 0$. Then$lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

2 Main Results

In this part, we will use implicit relation to show fixed point theorems.Definition 2.1 (Implicit Relation) Let $\boldsymbol{\Psi}$be the family of all real valued continuous functions $\boldsymbol{\psi}: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ non decreasing in the first argument for three variables. For some $\boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1})$, we consider the following conditions.

$(Imr1)$ For $x, y \in \mathbb{R}_+$, if $y \le \psi(x, x, y, \frac{x+y}{2})$, then $y \le \alpha x$,
$(Imr2)$ For $x \in \mathbb{R}_+$, if $y \le \psi(y, 0, 0, y)$, then $y = 0$ since $\alpha \in [0,1)$,
$(Imr3)$ For $x \in \mathbb{R}_+$, if $y \le \psi(0, 0, y, \frac{y}{2})$, then $y = 0$.

Theorem 2.2 Let $(\tilde{X}, p)$ be a complete partial metric space and let $\tilde{T}: \tilde{X} \to \tilde{X}$ be a mapping, such that

$$p(\tilde{T}\tilde{u}, \tilde{T}\tilde{v}) \le \psi\{p(\tilde{u}, \tilde{v}), p(\tilde{u}, \tilde{T}\tilde{u}), p(\tilde{v}, \tilde{T}\tilde{v}), \frac{p(\tilde{u}, \tilde{T}\tilde{v}) + p(\tilde{v}, \tilde{T}\tilde{u})}{2}\} \quad (2.1)$$

for all $\tilde{u}, \tilde{v} \in \tilde{X}$ and some $\tilde{\psi} \in \Psi$. If Ψ satisfies the condition $(Imr1), (Imr2)$ and $(Imr3)$, then $\tilde{T}$ has a unique fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in N$, put $u_{n+1} = \tilde{T}u_n$. It follows from equation (2.1) and $(pm4)$ that

$$\begin{aligned} p(u_n, u_{n+1}) = \quad & p(\tilde{T}u_{n-1}, \tilde{T}u_n) \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, \tilde{T}u_{n-1}), p(u_n, \tilde{T}u_n), \frac{p(u_{n-1}, \tilde{T}u_n) + p(u_n, \tilde{T}u_{n-1})}{2}\} \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1}, u_{n+1}) + p(u_n, u_n)}{2}\} \\ & \leq \psi\left\{\begin{matrix} p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \\ \frac{p(u_{n-1}, u_n) + p(u_n, u_{n+1}) - p(u_n, u_n) + p(u_n, u_n)}{2} \end{matrix}\right\} \end{aligned}$$

$$= \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1}, u_n) + p(u_n, u_{n+1})}{2}\} \tag{2.2}$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in [0,1)$, such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1). \tag{2.3}$$

Set $S_n = p(u_n, u_{n+1})$ and $S_{n_1} = p(u_{n-1}, u_n)$.

Then from equation (2.3), we obtain

$$S_n \leq \alpha S_{n-1} \leq \ldots \leq \alpha^n S_0.$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m, n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have:

$$\begin{aligned} & p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\ & -p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\ & \leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\ & = \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\ & = \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\ & \leq \alpha (\frac{1 - \alpha^{m-1}}{1 - \alpha}) S_0. \end{aligned}$$

In above inequality, taking the limits as $n, m \to \infty$, we have

$p(u_n, u_m) \to 0$ since $0 < \alpha < 1$,

hence, $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$ and this sequence is also Cauchy in $(\tilde{X}, d_p)$.

As $(\tilde{X}, p)$ is complete, therefore $(\tilde{X}, d_p)$ is also complete.

Thus by Lemma 1.5,

$$\lim_{n\to\infty} u_n \to w$$

$$p(w, w) = lim_{n\to\infty} p(w, u_n) = lim_{n\to\infty} p(u_n, u_m) = 0, \tag{2.4}$$

implies,

$$lim_{n\to\infty} d_p(w, u_n) = 0. \tag{2.5}$$

Now, we show that w is a fixed point of $\tilde{T}$.

From equation (2.4), we have

$p(w,w) = 0.$

By using inequality (2.1), we get

$$\begin{aligned} p(u_{n+1},\tilde{T}w) &= p(\tilde{T}u_n,\tilde{T}w) \\ &\leq \psi\{p(u_n,w),p(u_n,\tilde{T}u_n),p(w,\tilde{T}w),\frac{p(u_n,\tilde{T}w)+p(w,\tilde{T}u_n)}{2}\} \\ &= \psi\{p(u_n,w),p(u_n,u_{n+1}),p(w,\tilde{T}w),\frac{p(u_n,\tilde{T}w)+p(w,u_{n+1})}{2}\}. \end{aligned}$$

Note that $\psi \in \Psi$, then taking the limit as $n \to \infty$ and using equation (2.4), we get

$$p(u_{n+1},\tilde{T}w) \leq \psi\{0,0,p(w,\tilde{T}w),\tfrac{p(w,\tilde{T}w)}{2}\}.$$

Since ψ satisfies the condition $(Imr3)$, then

$$p(w,\tilde{T}w) = 0.$$

This implies $w = \tilde{T}w$.

Therefore, w is a fixed point of $\tilde{T}$.

Now we have to show that the fixed point of $\tilde{T}$ is a unique .

let w_1, w_2 be fixed points of $\tilde{T}$ with $w_1 \neq w_2$.

From equations (2.1) and (2.4)

$$\begin{aligned} p(w_1,w_2) &= p(\tilde{T}w_1,\tilde{T}w_2) \\ &\leq \psi\{p(w_1,w_2),p(w_1,\tilde{T}w_1),p(w_2,\tilde{T}w_2),\frac{p(w_1,\tilde{T}w_2),p(w_2,\tilde{T}w_1)}{2}\} \\ &= \psi\{p(w_1,w_2),p(w_1,w_1),p(w_2,w_2),\frac{p(w_1,w_2)+p(w_2,w_1)}{2}\} \\ &= \psi\{p(w_1,w_2),0,0,p(w_1,w_2)\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, we have

this implies, that $p(w_1,w_2) = 0$, since $0 < \alpha < 1$.

This implies that $w_1 = w_2$.

Thus the fixed point of $\tilde{T}$ is unique.

Theorem 2.3 Let $(\tilde{X},p)$ be a complete partial metric space and $\tilde{F},\tilde{G}$ be two self maps on $\tilde{X}$ satisfying the following:

$$p(\tilde{F}u,\tilde{G}v) \leq \psi\{p(u,v),p(u,\tilde{F}u),p(v,\tilde{G}v),\tfrac{p(u,\tilde{G}v)+p(v,\tilde{F}u)}{2}\} \tag{2.6}$$

for all $u,v \in \tilde{X}$ and some $\psi \in \Psi$. Then $\tilde{F}$ and $\tilde{G}$ have a unique common fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in \mathbb{N}$.

Put $u_{n+1} = \tilde{F}u_n$ and

$u_{n+2} = \tilde{G}u_{n+1}$ for $n = 0,1,2..$.

It follows from equation (2.5), Lemma 1.5 and $(pm4)$ that

$$\begin{aligned} p(u_n, u_{n+1}) &= p(\tilde{F}u_{n-1}, \tilde{G}u_n) \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, \tilde{F}u_{n-1}), p(u_n, \tilde{G}u_n), \frac{p(u_{n-1}, \tilde{G}u_n) + p(u_n, \tilde{F}u_{n-1})}{2}\} \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u-n+1), \frac{p(u_{n-1}, u_{n+1}) + p(u_n, u_n)}{2}\} \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1}, u_n)p(u_n, u_{n+1}) - p(u_n, u_n) + p(u_n, u_n)}{2}\} \\ &= \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1}, u_n)p(u_n, u_{n+1})}{2}\} \end{aligned}$$

$$p(u_n, u_{n+1}) = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1}, u_n)p(u_n, u_{n+1})}{2}\}. \quad (2.7)$$

Since ψ satisfies the condition $(Imr1)$, therefore there exists $\alpha \in [0,1)$ such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1) \quad (2.8)$$

Now we show that $\{u_n\}$ is a Cauchy sequence

Let $m, n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have

$$\begin{aligned} & p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\ & -p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\ & \leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\ & = \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\ & = \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\ & \leq \alpha (\frac{1 - \alpha^{m-1}}{1 - \alpha}) S_0. \end{aligned}$$

Assume that, $lim_{n\to\infty} u_n \to r$

by Lemma 1.5,

$$p(r, r) = lim_{n\to\infty} p(r, u_n) = lim_{n,m\to\infty} p(u_n, u_m) = 0. \quad (2.9)$$

This implies,

$$lim_{n\to\infty} d_p(r, u_n) = 0. \quad (2.10)$$

Now we have to prove that r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

$$\begin{aligned} p(u_{n+1}, \tilde{F}r) &= p(\tilde{F}u_n, \tilde{F}r) \\ &\leq \psi\{p(u_n, r), p(u_n, \tilde{F}u_n), p(r, \tilde{F}r), \frac{p(u_n, \tilde{F}r) + p(r, \tilde{F}u_n)}{2}\} \\ &= \psi\{p(u_n, r), p(u_n, u_{n+1}), p(r, \tilde{F}r), \frac{p(u_n, \tilde{F}r) + p(r, u_{n+1})}{2}\} \end{aligned}$$

Note that $\psi \in \Psi$, using (2.8), Lemma 1.6 and taking the limit as $n \to \infty$,

$$p(r, \tilde{F}r) \leq \psi\{0,0, p(r, \tilde{F}r), \frac{p(r, \tilde{F}r)}{2}\}$$

Since ψ satisfies the condition $(Imr3)$, then

$$p(r, \tilde{F}r) = 0.$$

This shows that $r = \tilde{F}r$ for all $r \in \tilde{X}$.

Similarly, we can show $r = \tilde{G}r$.

Thus r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

Now we will prove the uniqueness of fixed point of $\tilde{F}$ and $\tilde{G}$.

Let r' be another common fixed point of $\tilde{F}$ and $\tilde{G}$, that is, $\tilde{F}r' = \tilde{G}r' = r'$ with $r' \neq r$.

Thus we have to show to show that $r = r'$.

From equations (2.5) and (2.8)

$$\begin{aligned} p(r, r') = \quad & p(\tilde{F}r, \tilde{G}r') \\ & \leq \psi\{p(r, r'), p(r, \tilde{F}r), p(r', \tilde{G}r'), \frac{p(r, \tilde{G}r') + p(r', \tilde{F}r)}{2}\} \\ & = \psi\{p(r, r'), p(r, r), p(r', r'), \frac{p(r, r') + p(r', r)}{2}\} \\ & = \psi\{p(r, r'), 0, 0, p(r, r')\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$$p(r, r') = 0, \text{ since } 0 < \alpha < 1.$$

Thus, we get $r = r'$.

This shows that r is the unique common fixed point of $\tilde{F}$ and $\tilde{G}$.

Theorem 2.4 Let $(\tilde{X}, p)$ be a complete partial metric space and $\tilde{F}_1$, $\tilde{F}_2$ be continuous mappings on $\tilde{X}$ satisfying

$$p(\tilde{F_1}^m u, \tilde{F_1}^n v) \leq \psi\{p(u, v), p(u, \tilde{F_1}^m u), p(v, \tilde{F_2}^n v), \frac{p(u, \tilde{F_2}^n v) + p(v, \tilde{F_1}^m u)}{2} \tag{2.11}$$

for all $u, v \in \tilde{X}$, where m and n are some positive integers and some $\psi \in \Psi$. Then $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point in X.

Proof. Since $\tilde{F_1}^m$ and $\tilde{F_2}^n$ satisfy the condition of Theorem 2.2. So $\tilde{F_1}^m$ and $\tilde{F_2}^n$ have a unique common fixed point.

Then, we have

$$\begin{aligned} & \tilde{F_1}^m \tilde{z} = \tilde{z} \\ & \Rightarrow \tilde{F}_1(\tilde{F_1}^m \tilde{z}) = \tilde{F}_1 \tilde{z} \\ & \tilde{F_1}^m(\tilde{F}_1 \tilde{z}) = \tilde{F}_1 \tilde{z}. \end{aligned}$$

If $\tilde{F}_1\tilde{z} = \tilde{r}_0$, then $\tilde{F_1}^m\tilde{r}_0 = \tilde{r}_0$.

So, $\tilde{F}_1\tilde{z}$ is a point of $\tilde{F_1}^m$.

Similarly, $\tilde{F}_2(\tilde{F_2}^n\tilde{z}) = \tilde{F}_2\tilde{z}$.

Now using equation (2.11) and Lemma 1.5, we obtain

$$\begin{aligned} p(\tilde{z},\tilde{F}_1\tilde{z}) &= p(\tilde{F_1}^m\tilde{z},\tilde{F_1}^m(\tilde{F}_1\tilde{z})) \\ &\leq \{p(\tilde{z},\tilde{F}_1z),p(\tilde{F}_1z,\tilde{F_1}^m(\tilde{F}_1\tilde{z})),p(\tilde{z},\tilde{F_1}^m\tilde{z}),\frac{p(\tilde{z},\tilde{F_1}^m(\tilde{F}_1\tilde{z}))+p(\tilde{F}_1\tilde{z},\tilde{F_1}^m\tilde{z})}{2}\} \\ &= \psi\{p(\tilde{z},\tilde{F}_1\tilde{z}),p(\tilde{z},\tilde{z}),p(\tilde{F}_1z,\tilde{F}_1z),\frac{p(\tilde{z},\tilde{F}_1\tilde{z})+p(\tilde{F}_1\tilde{z},\tilde{z})}{2}\} \\ &= \psi\{p(\tilde{z},\tilde{F}_1\tilde{z}),0,0,p(\tilde{z},\tilde{F}_1\tilde{z})\} \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$p(\tilde{z},\tilde{F}_1\tilde{z})$, since $0 < \alpha < 1$.

Thus, we have $\tilde{z} = \tilde{F}_1\tilde{z}$ for all $\tilde{z} \in \tilde{X}$.

Similarly, we can show that $\tilde{z} = \tilde{F}_2\tilde{z}$.

This shows that $\tilde{z}$ is a common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$.

For uniqueness of $\tilde{z}$, let $\tilde{z}' \neq \tilde{z}$ be another common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$. Then clearly $\tilde{z}'$ is also a common fixed point of $\tilde{F_1}^m$ and $\tilde{F_2}^n$ which implies $\tilde{z}' = \tilde{z}$.

Hence $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point.

Theorem 2.5 Let $\tilde{u}_\gamma$ be a family of continuous mappings on a complete partial metric space $(\tilde{X}, p)$ satisfying

$$p(\tilde{u}_\gamma u,\tilde{u}_\beta v) \leq \psi\{p(u,v),p(u,\tilde{u}_\gamma u),p(v,\tilde{u}_\beta v),\frac{p(u,\tilde{u}_\beta v)+p(v,\tilde{u}_\gamma u)}{2}\} \tag{2.12}$$

for all $\gamma,\beta \in \Psi$ with $\gamma \neq \beta$ and $u,v \in \tilde{X}$. Then there exists a unique $\tilde{z} \in \tilde{X}$ satisfying $\tilde{u}_\gamma\tilde{z} = \tilde{z}$ for all $\gamma \in \Psi$.

Proof. For $u_0 \in \tilde{X}$, we define a sequence as follows:-

$u_{n+1} = \tilde{u}_\gamma u_n,\ u_{n+2} = \tilde{u}_\beta u_{n+1},$ n=0,1,2...

It follows from $(2.12), (pm4)$ and Lemma 1.5 that

$$\begin{aligned} p(u_n,u_{n+1}) = \quad & p(\tilde{u}_\gamma u_{n-1},\tilde{u}_\beta u_n) \\ &\leq \psi\{p(u_{n-1},u_n),p(u_{n-1},\tilde{u}_(\gamma)u_{n-1}),p(u_n,\tilde{u}_\beta u_n),\frac{p(u_{n-1},\tilde{u}_\beta u_n)+p(u_n,\tilde{u}_\gamma u_{n-1})}{2}\} \\ &= \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\} \\ &\leq \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\} \\ &= \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\} \end{aligned} \tag{2.13}$$

$$p(u_n,u_{n+1}) \leq \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\}.$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in (0,1)$ such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1) \tag{2.14}$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (2.14), we have

$$\begin{aligned}
& p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\
& -p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\
& \leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\
& = \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\
& = \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\
& \leq \alpha (\frac{1 - \alpha^{m-1}}{1 - \alpha}) S_0.
\end{aligned}$$

Taking $\lim_{n,m \to \infty}$ in the above inequality, we get

$p(u_n, u_m) \to 0$, since $0 < \alpha < 1$.
$u_n \to \tilde{r}$ as $n \to \infty$.

Moreover by Lemma 1.5,

$$p(\tilde{r}, \tilde{r}) = \lim_{n \to \infty} p(\tilde{r}, u_{2n}) = \lim_{n,m \to \infty} p(\tilde{u}_n, \tilde{u}_m) = 0 \tag{2.15}$$

implies,

$$\lim_{n \to \infty} d_p(\tilde{r}, u_n) = 0 \tag{2.16}$$

By the continuity of $\tilde{u}_\alpha$ and $\tilde{u}_\beta$, it is clear that

$$\tilde{u}_\gamma \tilde{r} = \tilde{u}_\beta \tilde{r} = \tilde{r}.$$

Therefore $\tilde{r}$ is a common fixed point of $\tilde{u}_\gamma$ for all $\gamma \in \Psi$.

To prove the uniqueness, let us consider the another common fixed point $\tilde{r}'$ of $\tilde{u}_\gamma$ and $\tilde{u}_\beta$ where

$$\tilde{r} \neq \tilde{r}'$$

from equations (2.12) and (2.15), we obtain

$$\begin{aligned}
p(\tilde{r}, \tilde{r}') & = p(\tilde{u}_\gamma \tilde{r}, \tilde{u}_\beta \tilde{r}') \\
& \leq \psi \left\{ p(\tilde{r}, \tilde{r}'), p(\tilde{r}, \tilde{U}_\gamma \tilde{r}), p(\tilde{r}', \tilde{U}_\beta), \frac{p(\tilde{r}, \tilde{u}_\beta \tilde{r}') + p(\tilde{r}', \tilde{u}_\gamma \tilde{r})}{2} \right\} \\
& = \psi \left\{ p(\tilde{r}, \tilde{r}'), p(\tilde{r}, \tilde{r}), p(\tilde{r}', \tilde{r}'), \frac{p(\tilde{r}, \tilde{r}') + p(\tilde{r}', \tilde{r})}{2} \right\} \\
& = \psi \{ p(\tilde{r}, \tilde{r}'), 0, 0, p(\tilde{r}, \tilde{r}') \}
\end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$p(\tilde{r}, \tilde{r}') = 0$, since $0 < \alpha < 1$.

Thus, we get $\tilde{r} = \tilde{r}'$ for all $\tilde{r} \in \tilde{X}$.

This shows that $\tilde{r}$ is a unique common fixed point of $\tilde{u}_\gamma$for all $\gamma \in \Psi$.

3 Conclusion

We have introduced the implicit relation on $\mathbb{R}_+{}^4$. We have established fixed point and common fixed point theorems on partial metric spaces using this implicit link.

Competing Interests

Authors have declared that no competing interests exist.

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