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A Fourth Order Improved Numerical Scheme for the Generalized Burgers—Huxley Equation

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Abstract

A fourth order finite-difference scheme in a two-time level recurrence relation is proposed for the numerical solution of the generalized Burgers-Huxley equation. The resulting nonlinear system, which is analyzed for stability, is solved using an improved predictor-corrector method. The efficiency of the proposed method is tested to the kink wave using both appropriate boundary values and conditions. The results arising from the experiments are compared with the relevant ones known in the available bibliography.

Keywords: Burgers-Huxley; Finite-Difference Method; Modified Predictor-Corrector

1. Introduction

A. Hodgkin and A. Huxley [1] proposed a model, known henceforth as the Huxley equation, in order to explain the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. The most general form of the Huxley equation, known as the generalized Burgers-Huxley equation (BgH) [2,3], has the form [4]

$$u_t + \alpha u^{\delta} u_x - u_{xx} = \beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right);\ 0 \le x \le 1,\ t > 0, \tag{1.1}$$

where $u = u(x,t)$ is a sufficiently often differentiable function, α a real parameter, $\beta \ge 0$, $\gamma \in (0,1)$ and $\delta > 0$. Equation (1.1), which models the interaction between reaction mechanisms, convection effects and diffusion transport, is the modified Burgers equation for $\beta = 0$ (see [5] and the references therein), is also the Huxley equation [1] for $\alpha = 0$, $\delta = 1$ and is the Fitzhugh-Nagoma equation [6] for $\alpha = 0$.

Many researchers have used various methods to solve the BgH equation. A theoretical study of the BgH equation was found in Wang *et al.* [4], while analytical solutions using various techniques in [7-11], etc., have been proposed. As far as the numerical methods are concerned among others the Adomian decomposition method was used by Ismail *et al.* [12] for the BgH and the Burgers-Fisher equation, and by Hashim *et al.* [13] for the BgH equation. Javidi [14] used the pseudospectral method, while Javidi [15], Javidi and Golbabai [16] the spectral collocation method. Batiha *et al.* [17] used the variational iteration method and Khattak [18] the collocation method with radial basis functions. Babolian and Saeidian [19] used the homotopy analysis method, etc.

The initial condition associated with Equation (1.1) will be

$$u(x,0) = f(x);\ 0 \le x \le 1. \tag{1.2}$$

Theoretical Solution

It is known [4] that Equation (1.1) has the following kink wave solution

$$u(x,t) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[k(x-ct)\right]\right\}^{1/\delta} \tag{1.3}$$

in which $k = \dfrac{\gamma\delta\left[-\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}\right]}{4(\delta+1)}$

and $c = \dfrac{\alpha\gamma}{\delta+1} - \dfrac{(1+\delta-\gamma)\left[-\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}\right]}{2(\delta+1)}$

are the wave number and the velocity respectively.

2. The numerical Method

2.1. Development of the Method

2.1.1. Grid and Solution Vector

To obtain numerical solutions the region $R = \{(x,t) \in [0 < x < 1] \times]0,T]\}$ with its boundary ∂R consisting of the lines $x=0$, $x=1$ and $t=0$ is covered with a rectangular mesh of points, G, with co-ordinates $(x,t) = (x_m, t_n) = (mh, n\ell)$ with $m = 0,1,\cdots,N+1$. The theoretical solution of Equation (1.1) at the typical mesh point (x_m, t_n) will be denoted by u_m^n and the relevant of an approximating difference scheme by U_m^n.

Let the solution vector at time level $t = t_n$ be

$$U(t) = U^n = \left[U_0^n, U_1^n, \cdots, U_{N+1}^n\right]^{\mathrm{T}}. \tag{2.1}$$

2.1.2. Boundaries

The following were used:

1) The space derivatives at the left boundary $x=0$ were replaced with second order finite-difference replacements of the form ([20] p. 17)

$$u_x\big|_{x=0} = \frac{1}{2h}\left(-3U_0^n + 4U_1^n - U_2^n\right) + \mathcal{O}\left(h^2\right) \text{ as } h \to 0, \tag{2.2}$$

$$u_{xx}\big|_{x=0} = \frac{1}{h^2}\left(2U_0^n - 5U_1^n + 4U_2^n - U_3^n\right) + \mathcal{O}\left(h^2\right) \text{ as } h \to 0 \tag{2.3}$$

and with analogous replacements to the right boundary $x=1$.

2) The boundary conditions

$$u(0,t) = g_0(t) \text{ and } u(1,t) = g_1(t) \tag{2.4}$$

$$u_x\big|_{x=0,1} = 0;\ t > 0, \tag{2.5}$$

were used, while at the other interior points of the grid G the well-known approximants based on the central-difference formulas.

2.2. The Proposed Method

Applying Equation (1.1) at each point of the grid G at time level $t = t_n = n\ell$; $n = 1,2,\cdots$ leads to a first-order initial-value problem, which is written in a matrix-vector form as

$$\begin{aligned} D\boldsymbol{U}(t) &= -\alpha\Delta A\boldsymbol{U}(t) + B\boldsymbol{U}(t) + \beta\tilde{\Delta}\boldsymbol{U}(t);\ t > 0 \\ \boldsymbol{U}^0 &= \boldsymbol{U}(0) = \left[f(x_0), f(x_1), \cdots, f(x_{N+1})\right]^{\mathrm{T}} = \boldsymbol{f} \end{aligned} \tag{2.6}$$

in which $D = \operatorname{diag}\{\mathrm{d}/\mathrm{d}t\}$,

$$\Delta = \Delta^n = \Delta(t) = \operatorname{diag}\left\{\Delta_m^n\right\} = \operatorname{diag}\left\{\left(U_m^n\right)^{\delta}\right\}, \tag{2.7}$$

$$\begin{aligned} \tilde{\Delta} = \tilde{\Delta}^n = \tilde{\Delta}(t) &= \operatorname{diag}\left\{\tilde{\Delta}_m^n\right\} \\ &= \operatorname{diag}\left\{\left[1 - \left(U_m^n\right]^{\delta}\right]\left[\left(U_m^n\right)^{\delta} - \gamma\right]\right\}; \end{aligned} \tag{2.8}$$

for $m = 0,1,\cdots,N+1$ are diagonal matrices,

$$A = \frac{1}{2h}\begin{bmatrix} -3 & 4 & -1 & & & \\ -1 & 0 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & -1 & 0 & 1 \\ & & & -1 & 4 & -3 \end{bmatrix}, \quad B = \frac{1}{h^2}\begin{bmatrix} 2 & -5 & 4 & -1 & & & \\ 1 & -2 & 1 & & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & & 1 & -2 & 1 \\ & & & -1 & 4 & -5 & 2 \end{bmatrix} \tag{2.9}$$

or

$$A = \frac{1}{2h}\begin{bmatrix} -2 & 2 & & & \\ -1 & 0 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 0 & 1 \\ & & & 2 & -2 \end{bmatrix}, \quad B = \frac{1}{h^2}\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix} \tag{2.10}$$

tridiagonal matrices arising from the use of the boundary values (2.2) - (2.3) or the boundary condition (2.5) respectively and $\boldsymbol{f}$ the vector of the initial condition, all of order $N+2$.

Relation (2.6) gives

$$D = -\alpha\Delta A + B + \beta\tilde{\Delta} \tag{2.11}$$

hence D^2 can easily be obtained.

Using the recurrence relation

$$\mathbf{U}(t+\ell) = \exp(\ell D)\mathbf{U}(t);\ t = 0, \ell, \cdots, \tag{2.12}$$

where $D\boldsymbol{U}(t)$ is given by (2.6) and replacing the matrix-exponential term with the fourth order rational approximant ([21] p. 134) gives

$$\begin{aligned} &\left(I - \frac{1}{2}\ell D + \frac{1}{12}\ell^2 D^2\right)\mathbf{U}(t+\ell) \\ &= \left(I + \frac{1}{2}\ell D + \frac{1}{12}\ell^2 D^2\right)\mathbf{U}(t). \end{aligned} \tag{2.13}$$

Equation (2.13) using the notations (2.7) - (2.8) and Equation (2.11) leads to the following nonlinear system

$$\mathbf{U}(t+\ell)-\frac{1}{2}\ell\left(-\alpha\Delta^{n+1}A+B+\beta\tilde{\Delta}^{n+1}\right)\mathbf{U}(t+\ell)$$
$$+\frac{1}{12}\ell^2\Big[\left(\alpha\Delta^{n+1}A\right)^2+B^2+\left(\beta\tilde{\Delta}^{n+1}\right)^2-\alpha\Delta^{n+1}AB$$
$$-\alpha\beta\Delta^{n+1}A\tilde{\Delta}^{n+1}-\alpha B\Delta^{n+1}A+\beta B\tilde{\Delta}^{n+1}$$
$$-\alpha\beta\tilde{\Delta}^{n+1}\Delta^{n+1}A+\beta\tilde{\Delta}^{n+1}B\Big]U(t) \tag{2.14}$$
$$=\mathbf{U}(t)+\frac{1}{2}\ell\left(-\alpha\Delta^{n}A+B+\beta\tilde{\Delta}^{n}\right)\mathbf{U}(t)$$
$$+\frac{1}{12}\ell^2\Big[\left(\alpha\Delta^{n}A\right)^2+B^2+\left(\beta\tilde{\Delta}^{n}\right)^2-\alpha\Delta^{n}AB$$
$$-\alpha\beta\Delta^{n}A\tilde{\Delta}^{n}-\alpha B\Delta^{n}A+\beta B\tilde{\Delta}^{n}$$
$$-\alpha\beta\tilde{\Delta}^{n}\Delta^{n}A+\beta\tilde{\Delta}^{n}B\Big]\mathbf{U}(t).$$

Let

$r_1=\ell\alpha/4h$, $r_2=\ell/2h^2$, $r_3=\ell\beta/2$,
$r_4=\ell^2\alpha^2/48h^2$, $r_5=\ell^2/12h^4$, $r_6=\ell^2\beta^2/12$,
$r_7=\ell^2\alpha/24h^3$, $r_8=\ell^2\alpha\beta/24h$ and $r_9=\ell^2\beta/12h^2$.

Equation (2.14), when applied to the general mesh point of the grid *G*, gives

$$U_m^{n+1}+r_1\Delta_m^{n+1}\left(-U_{m-1}^{n+1}+U_{m+1}^{n+1}\right)-r_2\left(U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}\right)$$
$$-r_3\tilde{\Delta}_m^{n+1}U_m^{n+1}+r_4\Delta_m^{n+1}\Big[\Delta_{m-1}^{n+1}U_{m-2}^{n+1}+\Delta_{m+1}^{n+1}U_{m+2}^{n+1}$$
$$-\left(\Delta_{m-1}^{n+1}+\Delta_{m+1}^{n+1}\right)U_m^{n+1}\Big]+r_5\Big(U_{m-2}^{n+1}-4U_{m-1}^{n+1}+6U_m^{n+1}$$
$$-4U_{m+1}^{n+1}+U_{m+2}^{n+1}\Big)+r_6\left(\tilde{\Delta}_m^{n+1}\right)^2U_m^{n+1}-r_7\Delta_m^{n+1}\Big(-U_{m-2}^{n+1}$$
$$+2U_{m-1}^{n+1}-2U_{m+1}^{n+1}+U_{m+2}^{n+1}\Big)-r_8\Delta_m^{n+1}\Big(-\tilde{\Delta}_{m-1}^{n+1}U_{m-1}^{n+1}$$
$$+\tilde{\Delta}_{m+1}^{n+1}U_{m+1}^{n+1}\Big)-r_7\Big[-\Delta_{m-1}^{n+1}U_{m-2}^{n+1}+2\Delta_m^{n+1}\left(U_{m-1}^{n+1}-U_{m+1}^{n+1}\right)$$
$$+\Delta_{m+1}^{n+1}U_{m+2}^{n+1}+\left(\Delta_{m-1}^{n+1}-\Delta_{m+1}^{n+1}\right)U_m^{n+1}\Big]+r_9\Big(\tilde{\Delta}_{m-1}^{n+1}U_{m-1}^{n+1}$$
$$-2\tilde{\Delta}_m^{n+1}U_m^{n+1}+\tilde{\Delta}_{m+1}^{n+1}U_{m+1}^{n+1}\Big)-r_8\Delta_m^{n+1}\tilde{\Delta}_m^{n+1}\left(-U_{m-1}^{n+1}+U_{m+1}^{n+1}\right)$$
$$+r_9\tilde{\Delta}_m^{n+1}\left(U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}\right)$$
$$=U_m^n-r_1\Delta_m^n\left(-U_{m-1}^n+U_{m+1}^n\right)+r_2\left(U_{m-1}^n-2U_m^n+U_{m+1}^n\right)$$
$$+r_3\tilde{\Delta}_m^nU_m^n+r_4\Delta_m^n\Big[\Delta_{m-1}^nU_{m-2}^n+\Delta_{m+1}^nU_{m+2}^n-\left(\Delta_{m-1}^n+\Delta_{m+1}^n\right)$$
$$\cdot U_m^n\Big]+r_5\left(U_{m-2}^n-4U_{m-1}^n+6U_m^n-4U_{m+1}^n+U_{m+2}^n\right)+r_6\left(\tilde{\Delta}_m^n\right)^2U_m^n$$
$$-r_7\Delta_m^n\left(-U_{m-2}^n+2U_{m-1}^n-2U_{m+1}^n+U_{m+2}^n\right)$$
$$-r_8\Delta_m^n\left(-\tilde{\Delta}_{m-1}^nU_{m-1}^n+\tilde{\Delta}_{m+1}^nU_{m+1}^n\right)-r_7\Big[-\Delta_{m-1}^nU_{m-2}^n$$
$$+2\Delta_m^n\left(U_{m-1}^n-U_{m+1}^n\right)+\Delta_{m+1}^nU_{m+2}^n+\left(\Delta_{m-1}^n-\Delta_{m+1}^n\right)U_m^n\Big]$$
$$+r_9\left(\tilde{\Delta}_{m-1}^nU_{m-1}^n-2\tilde{\Delta}_m^nU_m^n+\tilde{\Delta}_{m+1}^nU_{m+1}^n\right)-r_8\Delta_m^n\tilde{\Delta}_m^n\left(-U_{m-1}^n+U_{m+1}^n\right)$$
$$+r_9\tilde{\Delta}_m^n\left(U_{m-1}^n-2U_m^n+U_{m+1}^n\right);$$
$$m=0,1,\ldots,N+1. \tag{2.15}$$

Stability Analysis

Following the Fourier method of analysing stability ([21] p. 142) if $\xi=e^{\kappa\ell}$ is the amplification factor and $\tilde{U}_m^n$ the numerical value of U_m^n actually obtained, an error of the form $U_m^n-\tilde{U}_m^n=\xi^n e^{im\mu h}$; $i=\sqrt{-1}$ with κ a complex number and μ real is considered. Then Equation (2.15) leads to the following stability equation

$$\left[1-r_1\tilde{\Delta}_0+4r_3\sin^2\omega+2ir_2\Delta_0\sin2\omega\right]\xi$$
$$=1+2r_1\tilde{\Delta}_0+r_6\tilde{\Delta}_0^2-8\left(r_3+r_9\tilde{\Delta}_0\right)\sin^2\omega+16r_5\sin^4\omega$$
$$-4r_4\Delta_0^2\sin^22\omega+4i\Delta_0\Big[\left(-r_2+2r_7-r_8\tilde{\Delta}_0\right)\sin2\omega \tag{2.16}$$
$$-r_7\sin4\omega\Big]$$

where U_0 a typical value of U_m^{n+1}, U_m^n; $m=0,1,\cdots,N+1$ used for the linearization of the nonlinear terms, $\Delta_0=U_0^\delta$, $\tilde{\Delta}_0=\left(1-\Delta_0\right)\left(\Delta_0-\gamma\right)$ and $\omega=\beta h/2$ with $\omega\in\left[0,\pi/2\right]$. Equation (2.16) is of the form

$$\breve{A}\xi=\breve{B};\ \breve{A},\breve{B}\in\mathcal{C} \tag{2.17}$$

with $\mathcal{C}$ the set of the complex numbers, so the von Neumann necessary criterion for stability $|\xi|\le1$ will always be satisfied when

$$\left|\breve{B}\right|\le\left|\breve{A}\right|. \tag{2.18}$$

Inequality (2.18) for $\omega=0$ leads to

$$\left|1+2r_1\tilde{\Delta}_0+r_6\tilde{\Delta}_0^2\right|\le\left|1-r_1\tilde{\Delta}_0\right|, \tag{2.19}$$

which for $\beta=0$ holds, while for $\beta\ne0$ will be satisfied when

$$\ell\ll1. \tag{2.20}$$

If $\omega=\pi/2$, inequality (2.18) leads to

$$\left|1+16r_5+2r_1\tilde{\Delta}_0+r_6\tilde{\Delta}_0^2-8\left(r_3+r_9\tilde{\Delta}_0\right)\right|$$
$$\le\left|1+4r_3-r_1\tilde{\Delta}_0\right|,$$

which subject to (2.20) holds.

2.3. The Modified Predictor-Corrector Scheme

To avoid solving the nonlinear system (2.14) the following Modified Predictor-Corrector (MPC) scheme is proposed.

2.3.1. Predictor

$\hat{\mathbf{U}}(t+\ell)$ is evaluated from the reccurence relation (2.12) replacing the matrix-exponential term with the following explicit second order rational approximant

$$\hat{\mathbf{U}}(t+\ell)=\left(I+\ell D+\frac{1}{2}\ell^2D^2\right)\mathbf{U}(t)+\mathcal{O}\left(\ell^2\right)\ as\ \ell\to0. \tag{2.21}$$

Then Equation (2.21) subject to Equation (2.11) using again the notations (2.7) - (2.8) leads to

$$\begin{aligned}&\hat{\mathbf{U}}(t+\ell)\\&=\mathbf{U}(t)+\ell\left[-\alpha\Delta^n A+B+\beta\tilde{\Delta}^n\right]\mathbf{U}(t)\\&+\frac{\ell^2}{2}\Big[\left(\alpha\Delta^n A\right)^2+B^2+\left(\beta\tilde{\Delta}^n\right)^2-\alpha\Delta^n AB\\&-\alpha\beta\Delta^n A\tilde{\Delta}^n-\alpha B\Delta^n A+\beta B\tilde{\Delta}^n\\&-\alpha\beta\tilde{\Delta}^n\Delta^n A+\beta\tilde{\Delta}^n B\Big]\mathbf{U}(t).\end{aligned}\quad(2.22)$$

Let $p_1=\ell/h^2$, $p_2=\ell\alpha/2h$, $p_3=\ell^2\alpha^2/8h^2$, $p_4=\ell^2\beta^2/2$, $p_5=\ell^2/2h^4$, $p_6=\ell^2\alpha/4h^3$, $p_7=\ell^2\alpha\beta/4h$ and $p_8=\ell^2\alpha/4h^3$.

Equation (2.22), when applied to the general mesh point of the grid G, gives

$$\begin{aligned}\hat{U}_m^{n+1}&=U_m^n+\ell\beta\tilde{\Delta}_m^n U_m^n+p_1\left(U_{m-1}^n-2U_m^n+U_{m+1}^n\right)\\&-p_2\Delta_m^n\left(U_{m+1}^n-U_{m-1}^n\right)+p_3\Delta_m^n\Big[\Delta_{m-1}^n U_{m-2}^n\\&-\left(\Delta_{m-1}^n+\Delta_{m+1}^n\right)U_m^n+\Delta_{m+1}^n U_{m+2}^n\Big]+p_4\left(\tilde{\Delta}_m^n\right)^2U_m^n\\&+p_5\left(U_{m-2}^n-4U_{m-1}^n+6U_m^n-4U_{m+1}^n+U_{m+2}^n\right)\\&-p_6\Delta_m^n\left(-U_{m-2}^n+2U_{m-1}^n-2U_{m+1}^n+U_{m+2}^n\right)\\&-p_7\left(U_m^n\right)^\delta\left(\tilde{\Delta}_{m+1}^nU_{m+1}^n-\tilde{\Delta}_{m-1}^nU_{m-1}^n\right)\\&-p_6\Big[-\Delta_{m-1}^nU_{m-2}^n+2\Delta_m^nU_{m-1}^n-2\Delta_m^nU_{m+1}^n\\&+\left(\Delta_{m-1}^n-\Delta_{m+1}^n\right)U_m^n+\Delta_{m+1}^nU_{m+2}^n\Big]\\&+p_8\left(\tilde{\Delta}_{m-1}^nU_{m-1}^n-2\tilde{\Delta}_m^nU_m^n+\tilde{\Delta}_{m+1}^nU_{m+1}^n\right)\\&-p_7\Delta_m^n\tilde{\Delta}_m^n\left(U_{m+1}^n-U_{m-1}^n\right)\\&+p_8\tilde{\Delta}_m^n\left(U_{m-1}^n-2U_m^n+U_{m+1}^n\right);\\&\qquad m=0,1,\ldots,N+1.\end{aligned}\quad(2.23)$$

Stability Analysis

Following again the Fourier method of analysing stability Equation (2.23) leads to the following stability equation

$$\begin{aligned}\xi&=1+\ell\beta\tilde{\Delta}_0+p_4\tilde{\Delta}_0^2-4\left(p_1+2p_8\tilde{\Delta}_0\right)\sin^2\omega\\&+16p_5\sin^4\omega-4p_3\Delta_0^2\sin^22\omega+2i\Delta_0\\&\times\left[\left(-p_2+4p_6-2p_7\tilde{\Delta}_0\right)\sin2\omega-4p_6\sin4\omega\right],\end{aligned}\quad(2.24)$$

which is of the form (2.17) with $A=1$. Then condition (2.18) for $\omega=0$ leads to

$$\left|1+\ell\beta\tilde{\Delta}_0+\frac{1}{2}\ell^2\beta^2\tilde{\Delta}_0^2\right|\le1.\quad(2.25)$$

which for $\beta=0$ is obvious, while for $\beta\ne0$ is satisfied when the condition (2.20) holds. When $\omega=\pi/2$ condition (2.18) leads to

$$\left|1+\left(-\frac{4}{h^2}+\beta\tilde{\Delta}_0\right)\ell+\left(\frac{8}{h^4}-\frac{4}{h^2}\beta\tilde{\Delta}_0+\frac{1}{2}\beta^2\tilde{\Delta}_0^2\right)\ell^2\right|\le1,\quad(2.26)$$

which again subject to (2.20) is always satisfied.

2.3.2. Corrector

The corrector arises from Equation (2.13) as follows

$$\begin{aligned}\boldsymbol{U}(t+\ell)&=\left(\frac{1}{2}\ell D-\frac{1}{12}\ell^2D^2\right)\hat{\boldsymbol{U}}(t+\ell)\\&+\left(I+\frac{1}{2}\ell D+\frac{1}{12}\ell^2D^2\right)\boldsymbol{U}(t).\end{aligned}\quad(2.27)$$

Instead of the classical substitution of $\boldsymbol{U}(t+\ell)$ in the right-hand side of (2.27) by $\hat{\boldsymbol{U}}(t+\ell)$, a modified predictor-corrector method (MPC) was applied [5]. The MPC method, which is *explicit* and is applied *once*, consists of considering (2.27) component-wise and using an updated component in the corrector vector as soon as it becomes available. Hence, in computing U_m^{n+1} the corrected value U_{m-1}^{n+1} instead of the predicted value $\hat{U}_{m-1}^{n+1}$ is used. The stability analysis of the corrector is given in Section 2.2.1.

3. Numerical Results

For the linearization $U_0=\max_{m=0,1,\ldots,N+1}\{u_m^0\}$ was given. Let the error at time level $t=n\ell$; $n=1,2,\cdots$ be $e=e(t)=L_\infty=\max_{m=0,1,\cdots,N+1}\left|u_m^n-U_m^n\right|$ and x_e the x-coordinate at which e occurs. Then $e_{(2.2)\text{ - }(2.3)}$ denotes the error arising when using the boundary values (2.2) - (2.3), while analogous notations for the other boundary conditions are used. In all experiments the initial condition (1.2) was given by the value $f(x)=u(x,0)$ with u the theoretical solution (1.3). Experiments proved that the most accurate results are obtained for $h=0.1$ and $\ell=10^{-4}$. For reasons of comparison with the corresponding works in [12,13,16,17] the same parameter values were used.

3.1. Problem [12]

From the experiments the following are deduced:

1) when $\alpha\ne0$ (**Table 1**) using:

i) the boundary values (2.2) - (2.3) the method introduced gives more accurate results for all time levels used than the corresponding results in [12] and marginally more accurate than those in [13,17],

ii) the boundary condition (2.5) gives more accurate results than those in [12] and approximately equivalent to those in [13,17].

From (i) - (ii) it is deduced that the boundary values (2.2) - (2.3) give more accurate results than the boundary condition (2.5).

2) when $\alpha = 0$ (**Table 2**) using the boundary values (2.2) - (2.3) the method introduced has given

-for $\delta = 1$ more accurate results for all time levels used than the corresponding in [12], and

-for $\delta > 1$ results with marginally inferior accuracy to those in [12].

In **Figure 1(a)** the solution u for $t \in (0,1)$ and $x \in \left[-10^4, 10^4\right]$ is shown, while in **Figure 1(b)** the relevant solution U when $x \in [0,1]$.

Table 1. Problem [12]. Comparisons of the proposed method for various values of *x*, *t* with $\alpha = 1$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 1$ ($h = 0.1$, $\ell = 10^{-4}$).

t	x	Exact	$e_{(2.2)-(2.3)}$	$e_{(2.5)}$	e [12]	e [13]	e [17]
	0.1	0.5000187E−03	1.87406E−08	1.26463E−09	1.93715E−07	1.87406E−08	1.87405E−08
0.05	0.5	0.5000687E−03	1.87399E−08	1.97698E−08	1.93730E−07	1.87406E−08	1.87405E−08
	0.9	0.5001187E−03	1.87250E−08	4.60177E−08	1.93745E−07	1.87406E−08	1.87405E−08
	0.1	0.5000250E−03	3.74813E−08	6.39532E−09	3.87434E−07	3.74812E−08	3.74813E−08
0.1	0.5	0.5000750E−03	3.74736E−08	3.99558E−08	3.87464E−07	3.74812E−08	1.37481E−08
	0.9	0.5001250E−03	3.74186E−08	7.66328E−08	3.87494E−07	3.74812E−08	3.74813E−08
	0.1	0.5001374E−03	3.74814E−07	3.29223E−07	3.87501E−06	3.74812E−07	3.74812E−07
1	0.5	0.5001874E−03	3.72103E−07	3.79222E−07	3.87531E−06	3.74812E−07	3.74813E−07
	0.9	0.5002374E−03	3.68427E−07	4.29222E−07	3.87561E−06	3.74812E−07	3.74813E−07

Table 2. Problem [12]. Comparisons of the proposed method for various values of *x*, *t* and δ with $\alpha = 0$, $\beta = 1$ and $\gamma = 0.001$ ($h = 0.1$, $\ell = 10^{-4}$).

t	x	$\delta = 1$		$\delta = 2$		$\delta = 3$	
		$e_{(2.2)-(2.3)}$	e [12]	$e_{(2.2)-(2.3)}$	e [12]	$e_{(2.2)-(2.3)}$	e [12]
	0.1	2.49875E−08	1.87465E−07	1.11763E−06	5.58901E−07	3.96731E−06	1.9841E−06
0.05	0.5	2.49875E−08	1.87486E−07	1.11750E−06	5.58836E−07	3.96652E−06	1.98371E−06
	0.9	2.49874E−08	1.87508E−07	1.11737E−06	5.58772E−07	3.96572E−06	1.98331E−06
	0.1	4.99750E−08	3.74934E−07	2.23526E−06	1.11779E−06	7.93462E−06	3.96811E−06
0.1	0.5	4.99750E−08	3.74977E−07	2.23500E−06	1.11766E−06	7.93304E−06	3.96731E−06
	0.9	4.99749E−08	3.75019E−07	2.23474E−06	1.11753E−06	7.93144E−06	3.96652E−06
	0.1	4.99750E−07	3.75002E−06	2.23526E−05	1.11754E−05	7.93462E−05	3.96632E−05
1	0.5	4.99749E−07	3.75044E−06	2.23500E−05	1.00741E−05	7.93303E−05	3.96553E−05
	0.9	4.99749E−07	3.75086E−06	2.23474E−05	1.11728E−05	7.93143E−05	3.96473E−05

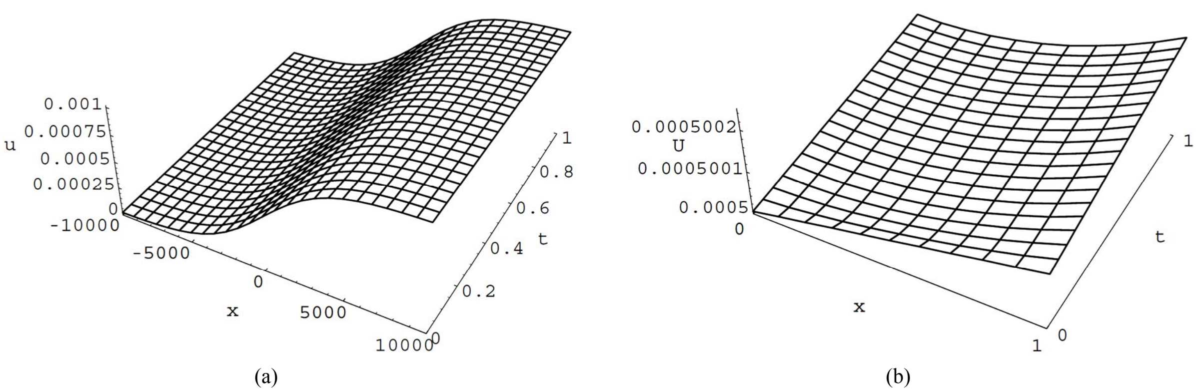

Figure 1. Problem [12] with $\delta = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.001$ when $t \in [0,1]$: In (a) the surface shows $u(x,t)$ for $x \in [-10^4, 10^4]$, while in (b) the numerical solution U when $x \in [0,1]$.

Table 3. Problem [17]. Comparisons of the proposed method for various values of δ and γ when $\alpha = \beta = 1$ ($h = 0.1$, $\ell = 10^{-4}$).

$t = 1$	$\delta = 1$	$\gamma = 10^{-3}$	$t = 0.5$	$\delta = 2$	$\gamma = 10^{-2}$	$t = 0.5$	$\delta = 4$	$\gamma = 10^{-2}$
x	$e_{(2.2)-(2.3)}$	e [17]	x	$e_{(2.2)-(2.3)}$	e [17]	x	$e_{(2.2)-(2.3)}$	e [17]
0.1	3.74814E−07	3.74812E−07	0.1	3.89463E−05	2.75734E−04	0.1	5.69322E−05	1.08762E−03
0.5	3.72103E−07	3.74814E−07	0.3	3.89656E−05	2.75614E−04	0.3	5.69778E−05	1.08644E−03
0.9	3.68427E−07	3.74813E−07	0.5	3.89844E−05	2.75493E−04	0.5	5.70134E−05	1.08527E−03

Table 4. Problem [16]. Boundary conditions (2.4) – (2.5). Comparisons of the proposed method for various values of t with$\alpha = 5$ and $\delta = 1$ ($h = 0.1$, $\ell = 10^{-4}$).

t	β	$\gamma = 10^{-3}$		$\gamma = 10^{-4}$		$\gamma = 10^{-5}$	
		Method	e [16]	Method	e [16]	e	e [16]
	1	3.1570E−08	3.1616E−08	3.1584E−10	3.1630E−10	3.3410E−12	3.1632E−12
0.3	10	3.9684E−07	3.9742E−07	3.9702E−09	3.9760E−09	3.9704E−11	3.9762E−11
	100	5.0291E−06	5.0365E−06	5.0316E−08	5.0389E−08	5.0318E−10	5.0392E−10
	1	3.3393E−08	3.3394E−08	3.3408E−10	3.3409E−10	3.3410E−12	3.3411E−12
0.9	10	4.1976E−07	4.1977E−07	4.1995E−09	4.1996E−09	4.1997E−11	4.1998E−11
	100	5.3165E−06	5.3166E−06	5.3221E−08	5.3223E−08	5.3224E−10	5.3225E−10

3.2. Problem [17]

From **Table 3** it is deduced that the method introduced using the boundary values (2.2) - (2.3) has given more accurate results for all time levels and parameters used than the relevant method in [17].

3.3. Problem [16]

For reasons of comparison with the relevant results in [18] the boundary conditions (2.4) - (2.5) with $g_0(t) = u(0,t)$ and $g_1(t) = u(1,t)$ were used. From **Table 4** it is deduced that the proposed method:

-has given marginally more accurate results to those in [16] for all time levels and β, γ used,

-for fixed α, δ and

- β, the accuracy increases and U tends to identify with u at long time level as γ is refined,
- γ, as β increases, the accuracy decreases.

4. Conclusions

An implicit finite difference scheme based on fourth-order rational approximants to the matrix exponential term was proposed for the numerical solution of the Burgers-Huxley equation. The resulting nonlinear scheme was solved using an improved predictor-corrector method. The computational efficiency of the proposed method given in detail in Section 3 was tested by comparing the numerical results to selected ones in [12,13,16,17] using both appropriate boundary values and conditions. Conclusions for the boundaries used were derived.

Since the real world problems lead to the numerical solution of nonlinear equations or systems of equations, the introduced low cost and easy-to-handle method enables us to obtain accurate solutions.

5. References

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