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# Positive Solutions of Singular Dirichlet Boundary Value Problems for Second Order Impulsive Differential Equations

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***Original Research Article***

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## Abstract

In this paper, we study the positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations. The existence of positive solutions is established by using the fixed point theorem in cones.

Keywords: Positive solution, singular two-point boundary value problem, second-order impulsive differential equations, fixed point theorem.
**MR (2000) Subject Classifications:** 34B15.

## 1 Introduction

Impulsive and singular differential equations play a very important role in modern applied mathematics due to their deep physical background and broad application. In this paper, we consider the existence of positive solutions of

$$\begin{cases} -x'' = f(t,x), & t \neq t_k, \quad 0 < t < 1, \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \cdots, m, \\ x(0) = x(1) = 0. \end{cases} \tag{1.1}$$

here $0 < t_1 < t_2 < \cdots < t_m < 1$ , J=[0,1] , $f \in C(J \times R^+, R^+)$ ,$I_k \in C(R^+, R^+), R^+ = [0, \infty)$ . $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ , $x'(t_k^+)$ (respectively $x'(t_k^-)$) denote the right limit (respectively left limit) of $x'(t)$ at $t = t_k$ and $f(t,x)$ may be singular at $x = 0$ .

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In recent years, boundary problems of second-order differential equations with impulses have been studied extensively in the literature (see for instance [1-9] and their references). In [1], Lin and Jiang studied the second-order impulsive differential equation with no singularity and obtained two positive solutions by using the fixed point index theorem in cones. However they did not consider the case when the function is singular. Motivated by the work mentioned above, we study the positive solutions of nonlinear singular two-point boundary value problems for second order impulsive differential equations (1.1) in this paper. Our argument is based on the fixed point theorem in cones.

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

$(H_1)$: There exists an $\varepsilon_0 > 0$ such that $f(t,x)$ and $I_k(x)$ are nonincreasing in $x \le \varepsilon_0$, for each fixed $t \in [0,1]$

$(H_2)$: For each fixed $0 < \theta \le \varepsilon_0$

$$0 < \int_0^1 f(s, \theta s(1-s))ds < \infty$$

$(H_3)$: $\varphi_1(t) = \sin \pi t$ is the eigenfunction related to the smallest eigenvalue $\lambda_1 = \pi^2$ of the eigenvalue problem $-\varphi'' = \lambda\varphi,\ \varphi(0) = \varphi(1) = 0.$

$(H_4)$: $f^{\infty} + \frac{\sum_{k=1}^{m} I^{\infty}(k)\varphi_1(t_k)}{\int_0^1 t(1-t)\varphi_1(t)dt} < \lambda_1,$

where $f^{\infty} = \lim_{x\to+\infty} \sup \max_{t\in[0,1]} \frac{f(t,x)}{x}, \quad I^{\infty}(k) = \lim_{x\to+\infty} \sup \frac{I_k(x)}{x}.$

**Theorem 1**. Assume that $(H_1)-(H_4)$ are satisfied. Then problem (1.1) has at least one positive solution $x$ .Moreover, there exists a $\theta^* > 0$ such that

$$x(t) \ge \theta^* t(1-t), \qquad t \in [0,1].$$

# 2 Preliminary

In order to define the solution of (1.1) we shall consider the following space.
Let $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$,

$$PC'(J,R) = \{x \in C(J,R); x'|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}),\ x'(t_k^-) = x'(t_k), \exists\, x'(t_k^+), k = 1,2,\cdots,m\}$$

With the norm $\|x\|_{PC'} = \max\{\|x\|, \|x'\|\}$, *here* $\|x\| = \sup_{t\in[0,1]} |x(t)|,\ \|x'\| = \sup_{t\in[0,1]} |x'(t)|$. Then $PC'(J,R)$ is a Banach space.

**Definition 2.1:** A function $x \in PC'(J,R) \cap C^2(J',R)$ is a solution of $(1.1)$ if it satisfies the differential equation

$$x'' + f(t,x) = 0, \quad t \in J'$$

and the function $x$ satisfies the conditions $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-) = -I_k(x(t_k))$, and the Dirichlet boundary conditions $x(0) = x(1) = 0$.

Lemma 2.1 [9]: If $x$ is a solution of the equation

$$x(t) = \int_0^1 G(t,s) f(s,x(s))ds + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)), \qquad t \in J \tag{2.1}$$

then $x$ is a solution of (1.1), where G(t,s) is the Green's function to the Dirichlet boundary value problem $-x'' = 0, x(0) = x(1) = 0, \quad and$

$$G(t,s) := \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases}$$

One can find that

$$t(1-t)G(s,s) \le G(t,s) \le G(s,s) = s(1-s), \quad (t,s) \in [0,1]\times[0,1]. \tag{2.2}$$

By using (2.1)and (2.2),we know that for every solution of problem(1.1) .One has

$$\|x\| \le \int_0^1 G(s,s) f(s,x(s))ds + \sum_{k=1}^{m} G(t_k,t_k) I_k(x(t_k)),$$

$$x(t) \ge t(1-t)\int_0^1 G(s,s) f(s,x(s))ds + t(1-t)\sum_{k=1}^{m} G(t_k,t_k) I_k(x(t_k))$$

$$\ge t(1-t)\|x\|, \quad t \in [0,1].$$

# 3 Main Results

**Lemma 3.1:**Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in $E$ ,and $\|\cdot\|$ be increasing with respect to $K$ .Also, $r$ , $R$ are constants with $0 < r < R$ . Suppose that $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ ( $\Omega_R = \{x \in E, \|x\| < R\}$ ) is a continuous, compact map and assume that the conditions are satisfied:

$(i)$ $\| Ax \| > x$, for $x \in \partial\Omega_r \bigcap K$

$(ii)$ $x \neq \mu A(x)$, for $\mu \in [0,1)$ and $x \in \partial\Omega_R \bigcap K$

Then $A$ has a fixed point in $K \bigcap \{x \in E : r \leq \|x\| \leq R\}$.

**Proof.** In applications below, we take $E = C(I, R)$ and define

$$K = \{u \in C(I,R) : u(x) \geq \sigma \|u\|, x \in [0,1]\}$$

One may readily verify that $K$ is a cone in $E$. Now, let $r > 0$ such that

$$r < min\{\varepsilon_0, \int_0^1 G(\frac{1}{2}, s) f(s, \varepsilon_0) ds + \sum_{k=1}^{m} G(\frac{1}{2}, t_k) I_k(\varepsilon_0)\} \tag{3.1}$$

and let $R > r$ be chosen large enough later.

Let us define an operator $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ by

$$(Ax)(t) = \int_0^1 G(t,s) f(s, x(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k)), t \in J.$$

First we show that $A$ is well defined.To see this, notice that if $x \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$ then $r \leq \|x\| \leq R$ and $x(t) \geq t(1-t)\|x\| \geq t(1-t)r, 0 \leq t \leq 1.$ Also notice by $(H_1)$ that

$$f(t, x(t)) \leq f(t, rt(1-t)), \qquad when\ 0 \leq x(t) \leq r,$$

and

$$f(t, x(t)) \leq \max_{r \leq x \leq R} \max_{0 \leq t \leq 1} f(t, x) \qquad when\ r \leq x(t) \leq R.$$

These inequalities with $(H_2)$ guarantee that $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ is well defined.

Next we show that $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ .If $x \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$ , then we have

$$\|Ax\| \leq \int_0^1 G(s,s) f(s, x(s)) ds + \sum_{k=1}^{m} G(t_k, t_k) I_k(x(t_k)),$$

$$(Ax)(t) \ge t(1-t)\int_0^1 G(s,s)f(s,x(s))ds + t(1-t)\sum_{k=1}^m G(t_k,t_k)I_k(x(t_k))$$

$$\ge t(1-t)\|Ax\|, \quad t\in[0,1].$$

i.e. $Ax \in K$ so $A:(\bar{\Omega}_R \setminus \Omega_r)\bigcap K \to K.$

It is clear that $A$ is continuous and completely continuous.

We now show that

$$\| Ax \| > \|x\|, \qquad for\ x\in \partial\Omega_r \bigcap K \tag{3.2}$$

To see that, let $x\in\partial\Omega_r\bigcap K$ ,then $\|x\| = r$ and $x(t)\ge t(1-t)r$ for $t\in[0,1]$. So by $(H_1)$ and $(3.1)$ we have

$$\begin{aligned}(Ax)(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2},s)f(s,x(s))ds + \sum_{k=1}^m G(\frac{1}{2},t_k)I_k(x(t_k))\\ &\ge \int_0^1 G(\frac{1}{2},s)f(s,r)ds + \sum_{k=1}^m G(\frac{1}{2},t_k)I_k(r)\\ &\ge \int_0^1 G(\frac{1}{2},s)f(s,\varepsilon_0)ds + \sum_{k=1}^m G(\frac{1}{2},t_k)I_k(\varepsilon_0)\\ &> r = \|x\|.\end{aligned}$$

so $(3.2)$ is satisfied.

On the other hand, from $(H_4)$ , there exist $0<\varepsilon<\lambda_1 - f^\infty$ and $H>r$ such that

$$(\lambda_1 - \varepsilon - f^\infty)\int_0^1 t(1-t)\varphi_1(t)dt > \sum_{k=1}^m (I^\infty(k)+\varepsilon)\varphi_1(t_k). \tag{3.3}$$

$$f(t,x)\le (f^\infty+\varepsilon)x, I_k(x)\le (I^\infty(k)+\varepsilon)x \quad \forall\ t\in[0,1], x\ge H.$$

Let $C = \max\limits_{r\le x\le H}\max\limits_{0\le t\le 1} f(t,x) + \sum_{k=1}^m \max\limits_{r\le x\le H} I_k(x),$ it is clear that

$$f(t,x)\le f(t,rt(1-t)) + C + (f^\infty+\varepsilon)x, I_k(x)\le I_k(rt(1-t)) + C + (I^\infty(k)+\varepsilon)x, \forall\ t\in[0,1], x\ge 0.$$

Next we show that if $R$ is large enough ,then $\mu Ax \neq x$ for any $x \in K \bigcap \partial\Omega_R$ and $0 \leq \mu < 1.$ If this is not true ,then there exist $x_0 \in K \bigcap \partial\Omega_R$ and $0 \leq \mu_0 < 1$ such that $\mu_0 A x_0 = x_0$. Thus $\| x_0 \| = R > r$ and $x_0(t) \geq t(1-t)R.$ Note that $x_0(t)$ satisfies

$$\begin{cases} x_0''(t) + \mu_0 f(t, x_0(t)) = 0, & t \in J', \\ -\Delta x_0'|_{t=t_k} = \mu_0 I_k(x_0(t_k)), & k = 1, 2, \cdots, m, \\ x_0(0) = x_0(1) = 0. \end{cases} \tag{3.4}$$

Multiply equation (3.4) by $\varphi_1(t)$ and integrate from 0 to 1, using integration by parts in the left side, notice that

$$\begin{aligned} \int_0^1 x_0''(t)\varphi_1(t)dt &= \int_0^{t_1} \varphi_1(t)dx_0'(t) + \sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}} \varphi_1(t)dx_0'(t) + \int_{t_m}^1 \varphi_1(t)dx_0'(t) \\ &= \varphi_1(t_1)x_0'(t_1 - 0) - \int_0^{t_1} x_0'(t)\varphi_1'(t)dt \\ &+ \sum_{k=1}^{m-1}[\varphi_1(t_{k+1})x_0'(t_{k+1} - 0) - \varphi_1(t_k)x_0'(t_k + 0) - \int_{t_k}^{t_{k+1}} x_0'(t)\varphi_1'(t)dt] \\ &- \varphi_1(t_m)x_0'(t_m + 0) - \int_{t_m}^1 \varphi_1'(t)x_0'(t)dt \\ &= -\sum_{k=1}^{m} \Delta x_0'(t_k)\varphi_1(t_k) - \int_0^1 \varphi_1'(t)x_0'(t)dt; \end{aligned}$$

also notice that

$$\int_0^1 \varphi_1'(t)x_0'(t)dt = \int_0^1 \varphi_1'(t)dx_0(t) = -\int_0^1 x_0(t)\varphi_1''(t)dt = \lambda_1\int_0^1 x_0(t)\varphi_1(t)dt,$$

thus

$$\begin{aligned} \int_0^1 x_0''(t)\varphi_1(t)dt &= -\sum_{k=1}^{m} \Delta x_0'(t_k)\varphi_1(t_k) - \lambda_1\int_0^1 x_0(t)\varphi_1(t)dt \\ &= \sum_{k=1}^{m} \mu_0 I_k(x_0(t_k))\varphi_1(t_k) - \lambda_1\int_0^1 x_0(t)\varphi_1(t)dt. \end{aligned}$$

So we obtain

$$\lambda_1\int_0^1 x_0(t)\varphi_1(t)dt = \mu_0\sum_{k=1}^m I_k(x_0(t_k))\varphi_1(t_k) + \mu_0\int_0^1 \varphi_1(t) f(t, x_0(t))dt$$
$$\leq \sum_{k=1}^m (I^\infty(k)+\varepsilon)\varphi_1(t_k)x_0(t_k) + C\sum_{k=1}^m \varphi_1(t_k) + \sum_{k=1}^m I_k(rt(1-t))\varphi_1(t_k)$$
$$+(f^\infty+\varepsilon)\int_0^1 \varphi_1(t)x_0(t)dt + C\int_0^1 \varphi_1(t)dt + \int_0^1 \varphi_1(t) f(t, rt(1-t))dt$$

Consequently, we obtain that

$$(\lambda_1 - f^\infty - \varepsilon)\int_0^1 x_0(t)\varphi_1(t)dt \leq \sum_{k=1}^m (I^\infty(k)+\varepsilon)\varphi_1(t_k)x_0(t_k) + \int_0^1 \varphi_1(t) f(t, rt(1-t))dt$$
$$+C(\sum_{k=1}^m \varphi_1(t_k) + \int_0^1 \varphi_1(t)dt) + \sum_{k=1}^m I_k(rt(1-t))\varphi_1(t_k)$$
$$\leq \|x_0\|\sum_{k=1}^m (I^\infty(k)+\varepsilon)\varphi_1(t_k) + \int_0^1 \varphi_1(t) f(t, rt(1-t))dt$$
$$+C(\sum_{k=1}^m \varphi_1(t_k) + \int_0^1 \varphi_1(t)dt) + \sum_{k=1}^m I_k(rt(1-t))\varphi_1(t_k)$$

We also have

$$\int_0^1 x_0(t)\varphi_1(t)dt \geq \| x_0 \| \int_0^1 t(1-t)\varphi_1(t)dt$$

Thus

$$\| x_0 \| \leq \frac{\int_0^1 \varphi_1(t) f(t, rt(1-t))dt + C(\sum_{k=1}^m \varphi_1(t_k) + \int_0^1 \varphi_1(t)dt) + \sum_{k=1}^m I_k(rt(1-t))\varphi_1(t_k)}{(\lambda_1 - f^\infty - \varepsilon)\int_0^1 t(1-t)\varphi_1(t)dt - \sum_{k=1}^m (I^\infty(k)+\varepsilon)\varphi_1(t_k)} =: \overline{R}$$

Let $R > max\{\overline{R}, H\}$, then for any $x \in K \bigcap \partial\Omega_R$ and $0 \leq \mu < 1$, we have $\mu Ax \neq x$. Hence all the assumptions of Lemma 3.1 are satisfied, $A$ has a fixed point $x$ in $K \bigcap \{x \in E : r \leq \| x \| \leq R\}$, $x(t) \geq t(1-t)r \quad \forall t \in [0,1]$. Let $\theta^* := r$, this complete the proof of Theorem 1.

## Competing Interests

Author has declared that no competing interests exist.

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