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# Attributes Reduction Based on Interval-Valued Fuzzy Rough Sets

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## Abstract

**Aims:** This paper provides a systematic study on attribute reduction with interval-valued fuzzy rough sets.
**Study Design:** The interval-valued fuzzy rough sets are an important improvement of traditional rough set model to deal with both fuzziness and vagueness in data which the traditional one cannot handle.
**Place and Duration of Study:** The existing researches on interval-valued fuzzy rough sets mainly focus on the establishment of lower and upper approximation operators by using constructive and axiomatic approaches. Less effort has been put on the attributes reduction of databases based on interval-valued fuzzy rough sets.
**Methodology:** After introducing some concepts and theorems of attributes reduction with interval-valued fuzzy rough sets, we study the structure of the attributes reduction with interval-valued fuzzy rough sets and present an algorithm by using discernibility matrix to find all the attributes reductions with interval-valued fuzzy rough sets.
**Results:** Finally, we propose an example to demonstrate our idea and method in this paper.
**Conclusion:** With these discussions we construct a basic foundation for attributes reduction based on interval-valued fuzzy rough sets.

Keywords: Interval-valued fuzzy rough sets, rough sets, attributes reduction, discernibility matrix.

## 1 Introduction

Rough set theory, originally proposed by Pawlak [1], can be regarded as an effective mathematical vehicle for dealing with imprecise and ambiguous data analysis. This theory has been demonstrated to have its usefulness and versatility in successfully solving a variety of problems [2-4]. The theory of rough sets deals with the approximation of an arbitrary subset of a universe by two definable subsets called lower and upper approximations. By using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be

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unraveled and expressed in the form of decision rules [5-8]. The concept of attributes reduction can be viewed as the strongest and most important results in rough sets theory to distinguish itself from other theories. Many studies of attributes reduction with rough sets could be found in the literature [9-20]. For example, Tsang and Chen et al. [21,22] discussed attribute reduction with covering rough sets. Tsang et al. [23] introduced formal concepts of attributes reduction with fuzzy rough sets and completely studied the structure of attributes reduction. They also developed an algorithm using discernibility matrix to compute all the attributes reductions. Wang et al. [24] provided a systematic study on attribute reduction with rough sets based on general binary relations.

Interval-valued fuzzy (IVF for short) sets [25,26] is a natural extensions of Zadeh's fuzzy sets [27], which were conceived independently to avoid some of the defects of fuzzy sets. IVF set theory emerges from the observation that in a lot of cases, no objective procedure is available to select the crisp membership degrees of elements in a fuzzy set. Hence, the grade of membership of an element in the universe of discourse belonging to an interval-valued fuzzy set is represented by an interval in $[0,1]$. The interval-valued fuzzy sets are more precise and flexible to model vagueness and uncertainty in practice than those of fuzzy sets. They have been applied to different research fields [28-39]. Recently, some authors extended rough set theory into IVF sets [40-42]. For example, Gong et al. [40] combined the interval-valued fuzzy sets and the rough sets, and studied the basic theory of the interval-valued rough fuzzy sets. Sun et al. [41] presented an interval-valued fuzzy rough set model by means of integrating the classical Pawlak rough set theory with the interval-valued fuzzy set theory, investigated knowledge reduction of the interval-valued fuzzy information system, and obtained some knowledge reduction theorems. Zhang et al. [42] proposed a general study of $(\mathcal{I},\mathcal{T})$-interval-valued fuzzy rough sets on two universes of discourse integrating the rough set theory with the interval-valued fuzzy set theory by constructive and axiomatic approaches.

It is well known that any generalization of traditional rough set theory should address two important theoretical issues. The first one is to present reasonable definitions of set approximation operators, and the second one is to develop reasonable algorithms for attributes reduction. It should be noted that the existing interval-valued fuzzy rough sets mainly pay attention to constructing approximation operators. The study for the attributes reduction of interval-valued fuzzy rough sets is still blank. It should be noted that the values of attributes could be denoted by interval-valued fuzzy sets [29,33,34,38]. It is hard to deal with such attributes for the traditional rough sets and fuzzy rough sets. In view of the requirement of possible applications and the complement of theoretical aspect of rough sets, it is interesting and important to construct the attributes reduction with interval-valued fuzzy rough sets. This paper systematically studies attribute reduction with interval-valued fuzzy rough sets. The structure of reduction is completely investigated and an algorithm using discernibility matrix to find all the attributes reductions is proposed.

The rest of the paper is structured as follows. Section 2 presents the fundamentals of Pawlak's rough sets, and reviews some basic notions of interval-valued fuzzy rough sets. In Section 3, we present the concept of attributes reduction with interval-valued fuzzy rough sets, and develop an algorithm using discernibility matrix to compute all the attributes reductions. An illustrated example is proposed in Section 4. Finally, some concluding remarks are presented in Section 5.

# 2 Preliminaries

## 2.1 Rough Sets Attributes Reduction

The following basic concepts about Pawlak's rough sets can be found in [1,14,23].

An information system is a pair $\mathbf{A}=(U,A)$, where $U=\{x_1,x_2,\cdots,x_n\}$ is a nonempty finite set of objects and $A=\{a_1,a_2,\cdots,a_m\}$ is a nonempty finite set of attributes. With every subset of attributes $B\subseteq A$ we associate a binary relation $\mathrm{IND}(B)$, called $B$-indiscernibility relation, and defined as $\mathrm{IND}(B)=\{(x,y)\in U\times U: a(x)=a(y),\forall a\in B\}$. $\mathrm{IND}(B)$ is obviously an equivalence relation and $\mathrm{IND}(B)=\bigcap_{a\in B}\mathrm{IND}(\{a\})$. By $[x]_B$ we denote the equivalence class of $\mathrm{IND}(B)$ including $x$. For any subset $X\subseteq U$, $\underline{B}(X)=\{x\in U:[x]_B\subseteq U\}$ and $\overline{B}(X)=\{x\in U:[x]_B\cap U\neq\varnothing\}$ are called $B$-lower and $B$-upper approximations of $X$ in $\mathbf{A}$, respectively.

By $M(\mathbf{A})$ we denote a $n\times n$ matrix $(c_{ij})$, called the discernibility matrix of $\mathbf{A}$, such that $c_{ij}=\{a\in A: a(x_i)\neq a(x_j)\}$ for $i,j=1,2,\cdots,n$. A discernibility function $f(\mathbf{A})$ for an information system $\mathbf{A}=(U,A)$ is a Boolean function of $m$ Boolean variables $\overline{a_1},\overline{a_2},\cdots,\overline{a_m}$ corresponding to the attributes $a_1,a_2,\cdots,a_m$, respectively, and defined as

$$f(\mathbf{A})(\overline{a_1},\overline{a_2},\cdots,\overline{a_m})=\wedge\{\vee(c_{ij}):1\leq j<i\leq n\}$$

where $\vee(c_{ij})$ is the disjunction of all variables $\overline{a}$ such that $a\in c_{ij}$.
An attribute $a\in B\subseteq A$ is superfluous in $B$ if $\mathrm{IND}(B)=\mathrm{IND}(B-\{a\})$, otherwise $a$ is indispensable in $B$.

The collection of all indispensable attributes in $A$ is called the core of $\mathbf{A}$. We say that $B\subseteq A$ is independent in $\mathbf{A}$ if every attribute in $B$ is indispensable in $B$. $B\subseteq A$ is called a reduction in $\mathbf{A}$ if $B$ is independent and $\mathrm{IND}(B)=\mathrm{IND}(A)$. The set of all the reductions in $\mathbf{A}$ is denoted as $Red(\mathbf{A})$. Let $g(\mathbf{A})$ be the reduced disjunctive form of $f(\mathbf{A})$ obtained from $f(\mathbf{A})$ by applying the multiplication and absorption laws, then there exist $l$ and $X_k\subseteq A$ for $k=1,2,\cdots,l$ such that $g(\mathbf{A})=(\wedge X_1)\vee(\wedge X_2)\vee\cdots\vee(\wedge X_l)$ where each element in $X_k$ appears only one time. We have $Red(\mathbf{A})=\{X_1,\cdots,X_l\}$.

A decision system is a pair $\mathbf{A}^*=(U,A\cup\{a^*\})$, where $a^*$ is the decision attribute, $A$ is condition attribute set. We say $a \in B \subseteq A$ is relatively dispensable in $B$ if $POS_B\left(a^*\right) = POS_{B-\{a\}}\left(a^*\right)$, otherwise $a$ is said to be relatively indispensable in $B$, where $POS_B\left(a^*\right)$ is the union of $B$-lower approximation of all the equivalence classes induced by $a^*$, i.e., $POS_B\left(a^*\right) = \bigcup_{X \in U/a^*} \underline{B}(X)$. If every attribute in $B$ is relatively indispensable in $B$, we say that $B \subseteq A$ is relatively independent in $\mathbf{A}^*$. $B \subseteq A$ is called a relative reduction in $\mathbf{A}^*$ if $B$ is relatively independent in $\mathbf{A}^*$ and $POS_B\left(a^*\right) = POS_A\left(a^*\right)$. The collection of all relatively indispensable attributes in $A$ is called the relative core of $\mathbf{A}^*$.

Suppose $M\left(\mathbf{A}^*\right) = \left(c_{ij}\right)$. We denote a matrix $\mathbf{M}\left(\mathbf{A}^*\right) = \left(\mathbf{c}_{ij}\right)$ in the following way:

(1) $\mathbf{c}_{ij} = c_{ij} - \left\{a^*\right\}$, if ( $a^* \in c_{ij}$ and $x_i, x_j \in POS_A\left(a^*\right)$) or $pos\left(x_i\right) \neq pos\left(x_j\right)$;
(2) $\mathbf{c}_{ij} = \varnothing$, otherwise.

Here $pos : U \to \{0,1\}$ is defined as $pos(x) = 1$ if and only if $x \in POS_A\left(a^*\right)$. All the relative reductions can be computed in an analogous way as reductions of $M(\mathbf{A})$.

## 2.2 Interval-valued Fuzzy Rough Sets

Throughout this paper, let $I$ be a closed unit interval, i.e., $I = [0,1]$. Let $[I] = \{[a,b] : a \leq b, a, b \in I\}$. For any $a \in I$, define $\overline{a} = [a,a]$. Let $U$ be an ordinary nonempty set, and $P(U)$ be the power set of $U$.

**Definition 2.1 [40,41].** If $a_i \in I, i \in J, J = \{1, 2, \cdots, m\}$, we define

$$\vee_{i \in J} a_i = \sup\{a_i : i \in J\},\ \wedge_{i \in J} a_i = \inf\{a_i : i \in J\},$$

$$\vee_{i \in J} [a_i, b_i] = [\vee_{i \in J} a_i, \vee_{i \in J} b_i],\ \wedge_{i \in J} [a_i, b_i] = [\wedge_{i \in J} a_i, \wedge_{i \in J} b_i].$$

In particular, for $[a_i, b_i] \in [I], i = 1, 2$, we define

$$[a_1, b_1] = [a_2, b_2] \text{ iff } a_1 = a_2, b_1 = b_2;$$

$$[a_1, b_1] \leq [a_2, b_2] \text{ iff } a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \text{ iff } [a_1, b_1] \leq [a_2, b_2],\ [a_1, b_1] \neq [a_2, b_2].$$

The complement of $[a_1, b_1]$ is denoted by $[a_1, b_1]^c = \overline{1} - [a_1, b_1] = [1 - b_1, 1 - a_1]$.

**Definition 2.2 [40,41].** The mapping $A: U \to [I]$ is called an interval-valued fuzzy set in $U$. All interval-valued fuzzy set on $U$ are denoted as $IVF(U \times U)$. If $A \in IVF(U \times U)$, let $A(x) = \left[A^-(x), A^+(x)\right]$, where $x \in U$, then two fuzzy sets $A^-: U \to I$, and $A^+: U \to I$ are called the lower fuzzy set and the upper fuzzy set about $A$, respectively.

Obviously, every fuzzy set $A$ can be identified with the interval-valued fuzzy set of the form $\{[A(x), A(x)] | x \in U\}$.

Let $U$ be a non-empty finite universe. A binary interval-valued fuzzy subset $R$ of $U \times U$ is called an interval-valued fuzzy relation in $U$.

Some basic operations on $IVF(U)$ are defined as follows:

$\forall A, B \in IVF(U)$,
$A \subseteq B$ if and only if (iff) $A^-(x) \le B^-(x)$ and $A^+(x) \le B^+(x)$ for all $x \in U$,
$A \supseteq B$ iff $B \subseteq A$,
$A = B$ iff $A \subseteq B$ and $B \subseteq A$, i.e., $A^-(x) = B^-(x)$ and $A^+(x) = B^+(x)$ for all $x \in U$,

$$(A \cap B)(x) = \left[A^-(x), A^+(x)\right] \wedge \left[B^-(x), B^+(x)\right] = \left[A^-(x) \wedge B^-(x), A^+(x) \wedge B^+(x)\right],$$

$$(A \cup B)(x) = \left[A^-(x), A^+(x)\right] \vee \left[B^-(x), B^+(x)\right] = \left[A^-(x) \vee B^-(x), A^+(x) \vee B^+(x)\right].$$

For $[\alpha, \beta] \in [I]$, $\overline{[\alpha, \beta]}$ will be denoted by the constant interval-valued fuzzy set: $\overline{[\alpha, \beta]}(x) = [\alpha, \beta]$ for all $x \in U$. The interval-valued fuzzy universe set is $U = \bar{1}$, and the interval-valued fuzzy empty set is $\varnothing = \bar{0}$.

**Definition 2.3 [41].** For the interval-valued fuzzy relation $R \in IVF(U \times U)$, we say that

(1) $R$ is reflexive if $R(x, x) = \bar{1}$ for all $x \in U$,
(2) $R$ is symmetric if for all $(x, y) \in U \times U$, $R(x, y) = R(y, x)$,
(3) $R$ is transitive if for all $(x, z) \in U \times U$, $R(x, z) \ge \vee_{y \in U} \left[R(x, y) \wedge R(y, z)\right]$.

If the fuzzy relation $R$ is reflexive, symmetric and transitive, then $R$ is an interval-valued fuzzy equivalence relation.

The similarity class $[x]_R$ (interval-valued fuzzy equivalence class) with $x \in U$ is an interval-valued fuzzy set on $U$ defined by $[x]_R(y) = R(x, y)$ for all $y \in U$.

The collection of all interval-valued fuzzy similarity classes can be denoted as $U/R$.

**Definition 2.4 [41].** Let $U$ be a nonempty finite universe and $R \in IVF(U \times U)$. $(U,R)$ is called an interval-valued fuzzy approximation space. For any $A \in IVF(U)$, the upper and lower approximations of $A$ about $(U,R)$, denote by $\overline{R}(A)$ and $\underline{R}(A)$, are two interval-valued fuzzy sets and are, respectively, defined as follows:

$$\forall x \in U,$$

$$\overline{R}(A)(x) = \max\left\{A(y) \wedge R(x,y) : y \in U\right\},$$

$$\underline{R}(A)(x) = \min\left\{A(y) \vee \left(\bar{1} - R(x,y)\right) : y \in U\right\}.$$

If for any $x \in U$ , $\overline{R}(A)(x) = \underline{R}(A)(x)$, then the interval-valued fuzzy set $A$ is definable about interval-valued fuzzy approximation space $(U,R)$. Or else the interval-valued fuzzy set $A$ is rough about the interval-valued fuzzy approximation space, and $A$ is called an interval-valued fuzzy rough set. Meanwhile, the mappings $\underline{R} : IF(U) \to IF(U)$ and $\overline{R} : IF(U) \to IF(U)$ are referred to as the lower interval-valued fuzzy rough approximation operator and upper interval-valued fuzzy rough approximation operator.

Clearly, the above definition implies equivalences of the following form:

$$\forall x \in U,$$

$$\begin{aligned}\overline{R}(A)(x) &= \vee_{y \in U}\left(A(y) \wedge R(x,y)\right)\\ &= \left[\vee_{y\in U}\left(A^-(y) \wedge R^-(x,y)\right), \vee_{y \in U}\left(A^+(y) \wedge R^+(x,y)\right)\right],\end{aligned}$$

$$\begin{aligned}\underline{R}(A)(x) &= \wedge_{y\in U}\left(A(y) \vee \left(\bar{1} - R(x,y)\right)\right)\\ &= \left[\wedge_{y\in U}\left(A^-(y) \vee \left(1 - R^+(x,y)\right)\right), \wedge_{y\in U}\left(A^+(y) \vee \left(1 - R^-(x,y)\right)\right)\right].\end{aligned}$$

**Theorem 2.1 [41].** Let $U$ be a nonempty and finite universe of discourse and $R, R_1, R_2 \in IVF(U \times U)$. Then the upper and lower approximation operators in Definition 2.4 satisfy the following properties:
$\forall A, B \in IVF(U)$,

(1) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,
(2) $\underline{R}(\sim A) = \sim \overline{R}(A)$, $\overline{R}(\sim A) = \sim \underline{R}(A)$,
(3) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,
(4) $A \subseteq B$, $\underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$,

(5) $\overline{R}\left(\overline{(\alpha,\beta)}\cap A\right)=\overline{(\alpha,\beta)}\cap\overline{R}(A)$, $\underline{R}\left(\overline{(\alpha,\beta)}\cup A\right)=\overline{(\alpha,\beta)}\cup\underline{R}(A)$,

(6) $R_1\subseteq R_2\Rightarrow\underline{R_1}(A)\supseteq\underline{R_2}(A)$, $R_1\subseteq R_2\Rightarrow\overline{R_1}(A)\subseteq\overline{R_2}(A)$.

# 3 Attributes Reduction Based on Interval-Valued Fuzzy Rough Sets

In this section we will define attribute reduction based on interval-valued fuzzy rough sets for interval-valued fuzzy decision system and propose some equivalence conditions to describe the structure of attribute reduction. We also develop an algorithm using discernibility matrix to compute all the attribute reductions.

Following the attributes with IVF values will be called IVF attributes [23]. For every IVF attribute, an IVF similarity relation can be employed to measure the similar degree between every pair of objects [23]. If we substitute every IVF attribute by its corresponding IVF similarity relation and substitute the decision attribute by its corresponding equivalence relation, we can get an IVF decision system consisting of three parts, a finite universe of discourse, a family of IVF similarity relations and a crisp equivalence relation [23]. Thus every dataset with IVF value conditional attributes and symbolic decision attribute can be expressed as an IVF decision system so that it is convenient to deal with by techniques of IF rough sets [23].

Two key problems must be solved before we define attribute reduction based on IVF rough sets [23]. One is what should be invariant after reduction [23]. We employ the idea in traditional rough sets of keeping the positive region of decision attribute invariant to define relative reduction with IVF rough sets; here the positive region of decision attribute will be defined as the union of lower approximations of decision classes [23]. Another problem is the selection of aggregation operator for several IVF similarity relations [23]. By Theorem 2.1(6), a smaller IVF similarity relation can provide more precise lower approximations, thus triangular $\mathrm{Min}$ is a reasonable selection of aggregation operator for several IVF similarity relations [23]. We can define attribute reduction for IVF decision system based on IVF rough sets with these discussions [23].

Suppose $U$ is a finite universe of discourse, $\mathbf{R}$ is a finite set of interval-valued fuzzy similarity relations called conditional attributes set, $D$ is an equivalence relation called decision attribute with symbolic values, then $(U,\mathbf{R}\cup D)$ is called an interval-valued fuzzy decision system. Denote $Sim(\mathbf{R})=\cap\{R:R\in\mathbf{R}\}$, then $Sim(\mathbf{R})$ is also an interval-valued fuzzy similarity relation. Suppose $[x]_D$ is the equivalence class with respect to $D$ for $x\in U$, then the positive region of $D$ relative to $Sim(\mathbf{R})$ is defined as $POS_{Sim(\mathbf{R})}(D)=\cup_{x\in U}\underline{Sim(\mathbf{R})}([x]_D)$. We say that $R$ is dispensable relative to $D$ in $\mathbf{R}$ if $POS_{Sim(\mathbf{R})}(D)=POS_{Sim(\mathbf{R}-\{R\})}(D)$, otherwise we will say $R$ is indispensable relative to $D$ in $\mathbf{R}$. The family $\mathbf{R}$ is independent relative to $D$ if each $R\in\mathbf{R}$ is indispensable relative to $D$ in $\mathbf{R}$; otherwise $\mathbf{R}$ is dependent relative to $D$. $\mathbf{P}\subseteq\mathbf{R}$ is an attributes reduction of relative to $D$ if $\mathbf{P}$ is independent relative to $D$ and $POS_{Sim(\mathbf{R})}(D)=POS_{Sim(\mathbf{P})}(D)$, for short we call $\mathbf{P}$ a relative reduction of $\mathbf{R}$. The collection of all the indispensable elements relative to $D$ in $\mathbf{R}$ is called the core of $\mathbf{R}$ relative to $D$, denoted as $Core(\mathbf{R})$. Similar to the result in traditional rough sets we have $Core(\mathbf{R}) = \bigcap Red(\mathbf{R})$, $Red(\mathbf{R})$ is the collection of all relative reductions of $\mathbf{R}$. Following we study under what conditions that $\mathbf{P} \subseteq \mathbf{R}$ could be a relative reduction of $\mathbf{R}$ [23].

We define an interval-valued fuzzy point $x_\lambda$ as $x_\lambda(z) = \begin{cases} \lambda, & z = x, \\ \bar{0}, & z \neq x. \end{cases}$, where $\lambda = [\lambda_1, \lambda_2] \in [I]$. It is obvious that $A = \bigcup_{\lambda \le A(x)} x_\lambda$ for any $A \in IVF(U)$. Define

$(x_\lambda)_R(z) = \begin{cases} \bar{0}, & 1 - R^+(x,z) \ge \lambda_1 \; or \; 1 - R^-(x,z) \ge \lambda_2 \\ \lambda, & 1 - R^+(x,z) < \lambda_1 \; and \; 1 - R^-(x,z) < \lambda_2. \end{cases}$, $z \in U$, for any $x \in U$ and $\lambda = [\lambda_1, \lambda_2] \in [I]$. Clearly, we have $x_\lambda \subseteq (x_\lambda)_R$.

**Theorem 3.1.** $\underline{R}(A) = \bigcup \{(x_\lambda)_R : (x_\lambda)_R \subseteq A, \lambda \in [I], x \in U\}$.

**Proof:** $\underline{R}(A) \supseteq \bigcup \{(x_\lambda)_R : (x_\lambda)_R \subseteq A, \lambda \in [I], x \in U\}$ is clear.

If $x_\lambda \subseteq \underline{R}(A)$, then we have $\wedge_{y \in U} \left(A^-(y) \vee \left(1 - R^+(x,y)\right)\right) \ge \lambda_1$, thus if $1 - R^+(x,y) < \lambda_1$, then $A^-(y) \ge \lambda_1$ must hold.

If $1 - R^+(x,z) \ge \lambda_1$, then $(x_\lambda)_R^-(z) = 0$, so $(x_\lambda)_R^-(z) \le \underline{R}(A)^-(z)$ holds.

Suppose $1 - R^+(x,z) < \lambda_1$. If $1 - R^+(y,z) < \lambda_1$, then
$1 - R^+(x,y) \le \vee \{1 - R^+(x,z), 1 - R^+(y,z)\} < \lambda_1$, this implies $A^-(y) \ge \lambda_1$. Thus, we have
$\underline{R}(A)^-(z) = \wedge_{y \in U} \left(A^-(y) \vee \left(1 - R^+(z,y)\right)\right) \ge \lambda_1 = (x_\lambda)_R^-(z)$ for $z \in U$ satisfying $1 - R^+(x,z) < \lambda_1$.

This implies if $x_\lambda \subseteq \underline{R}(A)$, then $(x_\lambda)_R^-(z) \le \underline{R}(A)^-(z)$ holds for all $z \in U$.

On the other hand, If $x_\lambda \subseteq \underline{R}(A)$, then we have $\wedge_{y \in U} \left(A^+(y) \vee \left(1 - R^-(x,y)\right)\right) \ge \lambda_2$, thus if $1 - R^-(x,y) < \lambda_2$, then $A^+(y) \ge \lambda_2$ must hold.

If $1 - R^-(x,z) \ge \lambda_2$, then $(x_\lambda)_R^+(z) = 0$, so $(x_\lambda)_R^+(z) \le \underline{R}(A)^+(z)$ holds.
Suppose $1 - R^-(x,z) < \lambda_2$. If $1 - R^-(y,z) < \lambda_2$, then
$1 - R^-(x,y) \le \vee \{1 - R^-(x,z), 1 - R^-(y,z)\} < \lambda_2$, this implies $A^+(y) \ge \lambda_2$. Thus, we have
$\underline{R}(A)^+(z) = \wedge_{y \in U} \left(A^+(y) \vee \left(1 - R^-(z,y)\right)\right) \ge \lambda_2 = (x_\lambda)_R^+(z)$ for $z \in U$ satisfying $1 - R^-(x,z) < \lambda_2$.
This implies if $x_\lambda \subseteq \underline{R}(A)$, then $(x_\lambda)_R^+(z) \le \underline{R}(A)^+(z)$ holds for all $z \in U$.

Therefore, if $x_\lambda \subseteq \underline{R}(A)$, then $(x_\lambda)_R \subseteq \underline{R}(A)$ holds. Namely, if $x_\lambda \subseteq \underline{R}(A)$, then $x_\lambda \subseteq (x_\lambda)_R \subseteq \underline{R}(A) \subseteq A$ holds.

Hence, we have $\underline{R}(A) = \bigcup \{(x_\lambda)_R : (x_\lambda)_R \subseteq A, \lambda \in L, x \in U\}$.

It is easy to prove that $\underline{R}((x_\lambda)_R) = (x_\lambda)_R$, so $\{(x_\lambda)_R : \lambda \in [I], x \in U\}$ can be employed as the basic granular set to compute lower approximation of interval-valued fuzzy sets.

**Proposition 3.1.** For any $x, y \in U$, if $(x_\lambda)_R \neq (y_\lambda)_R$, then $(x_\lambda)_R \cap (y_\lambda)_R = \varnothing$.

**Proof:** If $(x_\lambda)_R \cap (y_\lambda)_R \neq \varnothing$, then there exists $z \in U$ satisfying $(x_\lambda)_R(z) = (y_\lambda)_R(z) = \lambda$, i.e. This implies $1 - R^+(x,z) < \lambda_1$, $1 - R^-(x,z) < \lambda_2$, $1 - R^+(y,z) < \lambda_1$ and $1 - R^-(y,z) < \lambda_2$ hold. Thus, we have $1 - R^+(x,y) \leq \vee\{1 - R^+(x,z), 1 - R^+(y,z)\} < \lambda_1$ and $1 - R^-(x,y) \leq \vee\{1 - R^-(x,z), 1 - R^-(y,z)\} < \lambda_2$, hence, $y_\lambda \subseteq (x_\lambda)_R = \underline{R}((x_\lambda)_R)$ and $x_\lambda \subseteq (y_\lambda)_R = \underline{R}((y_\lambda)_R)$. By Theorem 3.1, we have $(x_\lambda)_R \subseteq \underline{R}((y_\lambda)_R) = (y_\lambda)_R$ and $(y_\lambda)_R \subseteq \underline{R}((x_\lambda)_R) = (x_\lambda)_R$, hence, $(x_\lambda)_R = (y_\lambda)_R$.

According to Theorem 3.1 and Proposition 3.1, we know that the properties of $(x_\lambda)_R$ is similar to the properties for equivalence classes of a crisp equivalence relation, so $(x_\lambda)_R$ can be employed as the equivalence class of $x_\lambda$.

**Proposition 3.2.** The following proposition holds.

$(x_\lambda)_{Sim(\mathbf{R})} = \bigcap_{R \in \mathbf{R}} (x_\lambda)_R$.

**Proof:** For every $z \in U$,

$$\begin{aligned}
(x_\lambda)_{Sim(\mathbf{R})}(z) = \lambda &\Leftrightarrow 1 - (Sim(\mathbf{R}))^+(x,z) < \lambda_1 \text{ and } 1 - (Sim(\mathbf{R}))^-(x,z) < \lambda_2 \\
&\Leftrightarrow \vee_{R \in \mathbf{R}} (1 - R^+(x,z)) < \lambda_1 \text{ and } \vee_{R \in \mathbf{R}} (1 - R^-(x,z)) < \lambda_2 \\
&\Leftrightarrow 1 - R^+(x,z) < \lambda_1 \text{ and } 1 - R^-(x,z) < \lambda_2 \text{ for any } R \in \mathbf{R} \\
&\Leftrightarrow (x_\lambda)_R(z) = \lambda \text{ for any } R \in \mathbf{R} \\
&\Leftrightarrow \bigcap_{R \in \mathbf{R}} (x_\lambda)_R(z) = \lambda.
\end{aligned}$$

We complete the proof.

The facts mentioned in Theorem 3.1, Propositions 3.1 and 3.2 are the key points to our following discussion on the structure of reduction.
Since $POS_{Sim(\mathbf{R})}(D) = \bigcup_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)$,

$POS_{Sim(\mathbf{R})}(D)(x) = \left[ \vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^-(x), \vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^+(x) \right]$ and $U$ is finite, we know $\vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^-(x)$ can get its max value at some $\underline{Sim(\mathbf{R})}([z_1]_D)^-(x)$, and $\vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^+(x)$ can get its max value at some $\underline{Sim(\mathbf{R})}([z_2]_D)^+(x)$. The following theorem implies $\vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^-(x)$ always can get its max value at some $\underline{Sim(\mathbf{R})}([x]_D)^-(x)$, and $\vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^+(x)$ always can get its max value at some $\underline{Sim(\mathbf{R})}([x]_D)^+(x)$.

**Theorem 3.2.** If $(x_\lambda)_{Sim(\mathbf{R})} \subseteq [z]_D$, then $(x_\lambda)_{Sim(\mathbf{R})} \subseteq [x]_D$.

**Proof.** $(x_\lambda)_{Sim(\mathbf{R})} \subseteq [z]_D$ implies $(x_\lambda)^-_{Sim(\mathbf{R})}(y) \le D(z, y)$ and $(x_\lambda)^+_{Sim(\mathbf{R})}(y) \le D(z, y)$ for each $y \in U$. Let $y = x$, and we have $\lambda_1 \le D(z, x)$ and $\lambda_2 \le D(z, x)$. So we have

$$(x_\lambda)^-_{Sim(\mathbf{R})}(y) \le (x_\lambda)^-_{Sim(\mathbf{R})}(y) \wedge \lambda_1 \le D(z, y) \wedge D(z, x) \le D(x, y) = [x]_D(y)$$

And

$$(x_\lambda)^+_{Sim(\mathbf{R})}(y) \le (x_\lambda)^+_{Sim(\mathbf{R})}(y) \wedge \lambda_2 \le D(z, y) \wedge D(z, x) \le D(x, y) = [x]_D(y)$$

which imply $(x_\lambda)_{Sim(\mathbf{R})} \subseteq [x]_D$.

If $\lambda = [\lambda_1, \lambda_2] = POS_{Sim(\mathbf{R})}(D)(x)$, then there exist $z_1, z_2 \in U$ such that
$\lambda_1 = \vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^-(x) = \underline{Sim(\mathbf{R})}([z_1]_D)^-(x)$ and
$\lambda_2 = \vee_{z \in U} \underline{Sim(\mathbf{R})}([z]_D)^+(x) = \underline{Sim(\mathbf{R})}([z_2]_D)^+(x)$ which imply
$(x_\lambda)^-_{Sim(\mathbf{R})}(y) \le D(z_1, y) = [z_1]_D(y)$ and $(x_\lambda)^+_{Sim(\mathbf{R})}(y) \le D(z_2, y) = [z_2]_D(y)$ for every $y \in U$, so
$(x_\lambda)^-_{Sim(\mathbf{R})}(y) \le D(x, y) = [x]_D(y)$ and $(x_\lambda)^+_{Sim(\mathbf{R})}(y) \le D(x, y) = [x]_D(y)$ for every $y \in U$.

Therefore $\underline{Sim(\mathbf{R})}([x]_D)^-(x) \ge \lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^+(x) \ge \lambda_2$. Namely,
$\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x)$. Keeping the positive region invariant after deleting attribute from $\mathbf{R}$ is equivalent to keeping $\underline{Sim(\mathbf{R})}([x]_D)(x)$ invariant for every $x \in U$. With this argument, we have the following theorems to characterize a relative reduction of $\mathbf{R}$.

**Theorem 3.3.** Suppose $\mathbf{P} \subseteq \mathbf{R}$, then $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$ if and only if $\mathbf{P}$ satisfies $(x_\lambda)_{Sim(\mathbf{P})} \subseteq [x]_D$ for $\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x)$, $x \in U$.

**Proof.** "$\Rightarrow$" If $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$, then $POS_{Sim(\mathbf{R})}(D) = POS_{Sim(\mathbf{P})}(D)$. By Theorem 3.2, we have $\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x) = \underline{Sim(\mathbf{P})}([x]_D)(x)$. This implies $(x_\lambda)^-_{Sim(\mathbf{P})}(y) \le [x]_D(y)$ and $(x_\lambda)^+_{Sim(\mathbf{P})}(y) \le [x]_D(y)$ for every $y \in U$. Hence, $(x_\lambda)_{Sim(\mathbf{P})} \subseteq [x]_D$ holds.

"$\Leftarrow$" Since $Sim(\mathbf{R}) \subseteq Sim(\mathbf{P})$, we have $\underline{Sim(\mathbf{R})}(A) \supseteq \underline{Sim(\mathbf{P})}(A)$ for any $A \in IF(U)$ which implies $\underline{Sim(\mathbf{R})}([x]_D)^-(x) \ge \underline{Sim(\mathbf{P})}([x]_D)^-(x)$ and $\underline{Sim(\mathbf{R})}([x]_D)^+(x) \ge \underline{Sim(\mathbf{P})}([x]_D)^+(x)$. If $(x_\lambda)_{Sim(\mathbf{P})} \subseteq [x]_D$, then $(x_\lambda)^-_{Sim(\mathbf{P})}(y) \le [x]_D(y)$ and $(x_\lambda)^+_{Sim(\mathbf{P})}(y) \le [x]_D(y)$ for every $y \in U$. By Theorem 3.1, this imply $(x_\lambda)^-(y) \le \underline{Sim(\mathbf{P})}([x]_D)^-(y)$ and $(x_\lambda)^+(y) \le \underline{Sim(\mathbf{P})}([x]_D)^+(y)$ for every $y \in U$. Letting $y = x$, then we have $\lambda_1 \le \underline{Sim(\mathbf{P})}([x]_D)^-(x)$ and $\lambda_2 \le \underline{Sim(\mathbf{P})}([x]_D)^+(x)$. Therefore, $\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x) = \underline{Sim(\mathbf{P})}([x]_D)(x)$ which implies $POS_{Sim(\mathbf{R})}(D) = POS_{Sim(\mathbf{P})}(D)$. Hence, $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$.

Since $\mathbf{P} \subseteq \mathbf{R}$, we have $(x_\lambda)_{Sim(\mathbf{P})} \supseteq (x_\lambda)_{Sim(\mathbf{R})}$. Theorem 3.3 implies that keeping $POS_{Sim(\mathbf{R})}(D) = POS_{Sim(\mathbf{P})}(D)$ is equivalent to keeping $(x_\lambda)_{Sim(\mathbf{P})} \subseteq [x]_D$ for every $x \in U$ and

$$\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x).$$

**Theorem 3.4.** The following two statements are equivalent.

(1) $\mathbf{P} \subseteq \mathbf{R}$ contains a relative reduction of $\mathbf{R}$.

(2) For every $x \in U$ and $\lambda = (\lambda_1, \lambda_2) = \underline{Sim(\mathbf{R})}([x]_D)(x)$. If $(y_\lambda)_{Sim(\mathbf{R})} \not\subset [x]_D$, then $1 - Sim(\mathbf{P})^+(x, y) \ge \lambda_1$ or $1 - Sim(\mathbf{P})^-(x, y) \ge \lambda_2$.

**Proof.** (1) $\Rightarrow$ (2) If $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$, then by Theorem 3.3, $(x_\lambda)_{Sim(\mathbf{P})} \subseteq [x]_D$ for $\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x)$, $x \in U$. If $(y_\lambda)_{Sim(\mathbf{R})} \not\subset [x]_D$, then $(y_\lambda)_{Sim(\mathbf{P})} \not\subset [x]_D$. This implies $(x_\lambda)_{Sim(\mathbf{P})} \ne (y_\lambda)_{Sim(\mathbf{P})}$, by Proposition 3.1, $(x_\lambda)_{Sim(\mathbf{P})} \cap (y_\lambda)_{Sim(\mathbf{P})} = \varnothing$, that is $1 - Sim(\mathbf{P})^+(x, y) \ge \lambda_1$ or $1 - Sim(\mathbf{P})^-(x, y) \ge \lambda_2$.

(2) $\Rightarrow$ (1) For every $x \in U$, $\lambda = [\lambda_1, \lambda_2] = \underline{Sim(\mathbf{R})}([x]_D)(x)$. If $(y_\lambda)_{Sim(\mathbf{R})} \not\subset [x]_D$, then $1 - Sim(\mathbf{P})^+(x, y) \ge \lambda_1$ or $1 - Sim(\mathbf{P})^-(x, y) \ge \lambda_2$. This implies $(x_\lambda)_{Sim(\mathbf{P})} \cap (y_\lambda)_{Sim(\mathbf{P})} = \varnothing$ (In fact, if $(x_\lambda)_{Sim(\mathbf{P})} \cap (y_\lambda)_{Sim(\mathbf{P})} \ne \varnothing$, there exists $z \in U$ such that $(x_\lambda)_{Sim(\mathbf{P})}(z) = (y_\lambda)_{Sim(\mathbf{P})}(z) = \lambda$. This

implies $1-Sim(\mathbf{P})^{+}(x,z)<\lambda_1$, $1-Sim(\mathbf{P})^{-}(x,z)<\lambda_2$, $1-Sim(\mathbf{P})^{+}(y,z)<\lambda_1$ and $1-Sim(\mathbf{P})^{-}(y,z)<\lambda_2$. Hence, $1-Sim(\mathbf{P})^{+}(x,y)\le\left[1-Sim(\mathbf{P})^{+}(x,z)\vee 1-Sim(\mathbf{P})^{+}(z,y)\right]<\lambda_1$ and $1-Sim(\mathbf{P})^{-}(x,y)\le\left[1-Sim(\mathbf{P})^{-}(x,z)\vee 1-Sim(\mathbf{P})^{-}(z,y)\right]<\lambda_2$. It is a contradiction.) Thus, we have $(x_\lambda)_{Sim(\mathbf{P})}\subseteq[x]_D$. By Theorem 3.3, $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$.
□

**Proposition 3.3.** $(y_\lambda)_{Sim(\mathbf{R})}\not\subset[x]_D$ if and only if $\underline{Sim(\mathbf{R})}([x]_D)^{-}(y)<\lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^{+}(y)<\lambda_2$.

**Proof.** $\Rightarrow$ If $(y_\lambda)_{Sim(\mathbf{R})}\not\subset[x]_D$, there exists $z\in U$ such that $(y_\lambda)_{Sim(\mathbf{R})}(z)=\lambda$ and $[x]_D(z)=\bar{0}$. $(y_\lambda)_{Sim(\mathbf{R})}(z)=\lambda$ implies $1-Sim(\mathbf{R})^{+}(y,z)<\lambda_1$ and $1-Sim(\mathbf{R})^{-}(y,z)<\lambda_2$. Therefore, we have $\left(1-Sim(\mathbf{R})^{+}(y,z)\right)\vee[x]_D(z)<\lambda_1$ and $\left(1-Sim(\mathbf{R})^{-}(y,z)\right)\vee[x]_D(z)<\lambda_2$ which imply $\underline{Sim(\mathbf{R})}([x]_D)^{-}(y)<\lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^{+}(y)<\lambda_2$.

$\Leftarrow$ Suppose $(y_\lambda)_{Sim(\mathbf{R})}\subseteq[x]_D$. By Theorem 3.1, we have $(y_\lambda)_{Sim(\mathbf{R})}\subseteq\underline{Sim(\mathbf{R})}([x]_D)$. Therefore, $\underline{Sim(\mathbf{R})}([x]_D)^{-}(y)\ge\lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^{+}(y)\ge\lambda_2$. It is a contradiction.

Theorem 3.3 and 3.4 show that keeping $(x_\lambda)_{Sim(\mathbf{P})}\subseteq[x]_D$ for $\lambda=(\lambda_1,\lambda_2)=\underline{Sim(\mathbf{R})}([x]_D)(x)$ is equivalent to keeping $1-Sim(\mathbf{P})^{+}(x,y)\ge\lambda_1$ or $1-Sim(\mathbf{P})^{-}(x,y)\ge\lambda_2$ for $(y_\lambda)_{Sim(\mathbf{R})}\not\subset[x]_D$, and is equivalent to keeping $(x_\lambda)_{Sim(\mathbf{P})}\cap(y_\lambda)_{Sim(\mathbf{P})}=\varnothing$ for $(y_\lambda)_{Sim(\mathbf{R})}\not\subset[x]_D$, and is equivalent to keeping $1-Sim(\mathbf{P})^{+}(x,y)\ge\lambda_1$ or $1-Sim(\mathbf{P})^{-}(x,y)\ge\lambda_2$ for $\underline{Sim(\mathbf{R})}([x]_D)^{-}(y)<\lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^{+}(y)<\lambda_2$.

The statement of keeping $1-Sim(\mathbf{P})^{+}(x,y)\ge\lambda_1$ or $1-Sim(\mathbf{P})^{-}(x,y)\ge\lambda_2$ for $(y_\lambda)_{Sim(\mathbf{R})}\not\subset[x]_D$ (i.e. keeping $1-Sim(\mathbf{P})^{+}(x,y)\ge\lambda_1$ or $1-Sim(\mathbf{P})^{-}(x,y)\ge\lambda_2$ for $\underline{Sim(\mathbf{R})}([x]_D)^{-}(y)<\lambda_1$ and $\underline{Sim(\mathbf{R})}([x]_D)^{+}(y)<\lambda_2$) can easily be applied to design an algorithm to compute relative reductions.

**Proposition 3.4.** $\mathbf{P}\subseteq\mathbf{R}$ is a relative reduction of $\mathbf{R}$ if and only if $\mathbf{P}$ is the minimal subset of $\mathbf{R}$ satisfying the conditions in Theorems 3.3 and 3.4.

With the above discussion, we can develop an algorithm to compute the relative reductions. Suppose $U=\{x_1,\cdots,x_n\}$. We denote a $n\times n$ matrix $(c_{ij})$ by $M_D(U,\mathbf{R})$, called the discernibility matrix of $(U,\mathbf{R}\cup D)$, such that

(1) $c_{ij}=\{R:1-R^+(x_i,x_j)\geq\lambda_{i1}\ or\ 1-R^-(x_i,x_j)\geq\lambda_{i2}\}$, $\lambda_i=[\lambda_{i1},\lambda_{i2}]=\underline{Sim(\mathbf{R})}([x_i]_D)(x_i)$, $\lambda_j=[\lambda_{j1},\lambda_{j2}]=\underline{Sim(\mathbf{R})}([x_i]_D)(x_j)$ if $\lambda_{j1}<\lambda_{i1}$ and $\lambda_{j2}<\lambda_{i2}$;

(2) $c_{ij}=\varnothing$, otherwise.

$M_D(U,\mathbf{R})$ may not be symmetric and $c_{ii}=\varnothing$. $R\in c_{ij}\Leftrightarrow\left((x_i)_{\lambda_i}\right)_R\cap\left((x_j)_{\lambda_i}\right)_R=\varnothing\Leftrightarrow\forall z\in U$, $1-R^+(x_i,z)\geq\lambda_{i1}$ or $1-R^-(x_i,z)\geq\lambda_{i2}$ or $1-R^+(x_j,z)\geq\lambda_{i1}$ or $1-R^-(x_j,z)\geq\lambda_{i2}\Rightarrow\forall z\in U$, $1-Sim(\mathbf{R})^+(x_i,z)\geq\lambda_{i1}$ or $1-Sim(\mathbf{R})^-(x_i,z)\geq\lambda_{i2}$ or $1-Sim(\mathbf{R})^+(x_j,z)\geq\lambda_{i1}$ or $1-Sim(\mathbf{R})^-(x_j,z)\geq\lambda_{i2}\Leftrightarrow\left((x_i)_{\lambda_i}\right)_{Sim(\mathbf{R})}\cap\left((x_j)_{\lambda_i}\right)_{Sim(\mathbf{R})}=\varnothing$. Suppose $\mathbf{P}\subseteq\mathbf{R}$ contains a relative reduction of $\mathbf{R}$, then $\left((x_i)_{\lambda_i}\right)_{Sim(\mathbf{P})}\cap\left((x_j)_{\lambda_i}\right)_{Sim(\mathbf{P})}=\varnothing$ for $\lambda_{j1}\leq\lambda_{i1}$ and $\lambda_{j2}\leq\lambda_{i2}$. This implies that there exists $R\in\mathbf{P}$ such that $\left((x_i)_{\lambda_i}\right)_R\cap\left((x_j)_{\lambda_i}\right)_R=\varnothing$. This is equivalent to $\mathbf{P}$ containing an element in $c_{ij}$. Thus $c_{ij}$ is the collection of attributes which can keep $\left((x_i)_{\lambda_i}\right)_{Sim(\mathbf{R})}\cap\left((x_j)_{\lambda_i}\right)_{Sim(\mathbf{R})}=\varnothing$ for $\lambda_{j1}<\lambda_{i1}$ and $\lambda_{j2}<\lambda_{i2}$.

A discernibility function $f_D(U,\mathbf{R})$ for $(U,\mathbf{R}\cup D)$ is a Boolean function of $m$ Boolean variables $m$ $\overline{R_1},\overline{R_2},\cdots,\overline{R_m}$ corresponding to the interval-valued fuzzy attributes $R_1,R_2,\cdots,R_m$, respectively, and is defined as follows:

$$f_D(U,\mathbf{R})\left(\overline{R_1},\overline{R_2},\cdots,\overline{R_m}\right)=\wedge\left\{\vee(c_{ij}):c_{ij}\neq\varnothing\right\},$$

where $\vee(c_{ij})$ is the disjunction of all variables $\overline{R}$ such that $R\in c_{ij}$. In the sequel, $\overline{R_i}$ is simply denoted without ambiguity as $R_i$.

We have the following theorem for the relative core.

**Theorem 3.5.** $Core_D(\mathbf{R})=\{R\in\mathbf{R}:c_{ij}=\{R\},1\leq i,j\leq n\}$.

**Proof.** $R\in Core_D(\mathbf{R})\Leftrightarrow POS_{Sim(\mathbf{R})}(D)\neq POS_{Sim(\mathbf{R}-\{R\})}(D)\Leftrightarrow$ There exists $x_i\in U$ such that $\left((x_i)_{\lambda_i}\right)_{Sim(\mathbf{P})}\not\subset[x_i]_D$ for $\lambda_i=[\lambda_{i1},\lambda_{i2}]=\underline{Sim(\mathbf{R})}([x_i]_D)(x_i)$, and there also exists $x_j\in U$ such that $\left((x_j)_{\lambda_i}\right)_{Sim(\mathbf{R})}\not\subset[x_i]_D$ and $\left((x_i)_{\lambda_i}\right)_{Sim(\mathbf{R}\text{-}\{R\})}=\left((x_j)_{\lambda_i}\right)_{Sim(\mathbf{R}\text{-}\{R\})}$. $\Leftrightarrow$ $1-Sim(\mathbf{R}\text{-}\{R\})^+(x_i,x_j)<\lambda_{i1}$,

$1-Sim(\mathbf{R}\text{-}\{R\})^{-}(x_i,x_j)<\lambda_{i2}$ , and $\left(\left(x_i\right)_{\lambda_i}\right)_R \neq \left(\left(x_j\right)_{\lambda_i}\right)_R$ . $\Leftrightarrow$ $1-(R')^{+}(x_i,x_j)<\lambda_{i1}$ and $1-(R')^{-}(x_i,x_j)<\lambda_{i2}$ for any $R'\neq R$ , and $\left(\left(x_i\right)_{\lambda_i}\right)_R \neq \left(\left(x_j\right)_{\lambda_i}\right)_R$ . $\Leftrightarrow$ $1-(R')^{+}(x_i,x_j)<\lambda_{i1}$ and $1-(R')^{-}(x_i,x_j)<\lambda_{i2}$ for any $R'\neq R$ , and $1-R^{+}(x_i,x_j)\geq\lambda_{i1}$ or $1-R^{-}(x_i,x_j)\geq\lambda_{i2}$ . $\Leftrightarrow$ $c_{ij}=\{R\}$ .

The statement $c_{ij}=\{R\}$ implies that $R$ is the unique attribute to maintain

$\left(\left(x_i\right)_{\lambda_i}\right)_{Sim(\mathbf{R})} \cap \left(\left(x_j\right)_{\lambda_i}\right)_{Sim(\mathbf{R})} = \varnothing$ if $\lambda_{j1}<\lambda_{i1}$ and $\lambda_{j2}<\lambda_{i2}$ .

**Theorem 3.6.** If $\mathbf{P}\subseteq\mathbf{R}$ , then $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$ if and only if $\mathbf{P}\cap c_{ij}\neq\varnothing$ for every $c_{ij}\neq\varnothing$ .

**Proof.** $\Rightarrow$ Suppose that $\exists i_0,j_0\leq n$ , $c_{i_0j_0}\neq\varnothing$ , but $\mathbf{P}\cap c_{i_0j_0}=\varnothing$ . Namely, $\forall R\in\mathbf{P}$ , $R\notin c_{i_0j_0}$ which implies $1-R^{+}(x_{i_0},x_{j_0})<\lambda_{i_01}$ and $1-R^{-}(x_{i_0},x_{j_0})<\lambda_{i_02}$ for $\lambda_{j_01}\leq\lambda_{i_01}$ and $\lambda_{j_02}\leq\lambda_{i_02}$ . Hence, $1-Sim(\mathbf{P})^{+}(x_{i_0},x_{j_0})<\lambda_{i_01}$ and $1-Sim(\mathbf{P})^{-}(x_{i_0},x_{j_0})<\lambda_{i_02}$ for $\lambda_{j_01}\leq\lambda_{i_01}$ and $\lambda_{j_02}\leq\lambda_{i_02}$ . This is a contradiction to the condition that $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$ .

$\Leftarrow$ $\forall R\in\mathbf{P}\cap c_{ij}$ . According to Definition of $c_{ij}$ , we have $1-R^{+}(x_i,x_j)\geq\lambda_{i1}$ or $1-R^{-}(x_i,x_j)\geq\lambda_{i2}$ for $\lambda_{j1}\leq\lambda_{i1}$ and $\lambda_{j2}\leq\lambda_{i2}$ . Therefore, $1-Sim(\mathbf{P})^{+}(x_i,x_j)\geq 1-R^{+}(x_i,x_j)\geq\lambda_{i1}$ or $1-Sim(\mathbf{P})^{-}(x_i,x_j)\geq 1-R^{-}(x_i,x_j)\geq\lambda_{i2}$ for $\lambda_{j1}\leq\lambda_{i1}$ and $\lambda_{j2}\leq\lambda_{i2}$ . By Theorem 3.4, $\mathbf{P}$ contains a relative reduction of $\mathbf{R}$ .

**Corollary 3.1.** Suppose $\mathbf{P}\subseteq\mathbf{R}$ , then $\mathbf{P}$ is a relative reduction of $\mathbf{R}$ if and only if $\mathbf{P}$ is the minimal set satisfying $\mathbf{P}\cap c_{ij}\neq\varnothing$ for every $c_{ij}\neq\varnothing$ .

Let $g_D(U,\mathbf{R})$ be the reduced disjunctive form of $f_D(U,\mathbf{R})$ obtained from $f_D(U,\mathbf{R})$ by applying the multiplication and absorption laws, then there exist $l$ and $\mathbf{R}_k\subseteq\mathbf{R}$ for $k=1,2,\cdots,l$ such that $g_D(U,\mathbf{R})=(\wedge\mathbf{R}_1)\vee(\wedge\mathbf{R}_2)\vee\cdots\vee(\wedge\mathbf{R}_l)$ where each element in $\mathbf{R}_k$ appears only one time. We have the following theorem.

**Theorem 3.7.** $Red_D(\mathbf{R})=\{\mathbf{R}_1,\cdots,\mathbf{R}_l\}$ .

**Proof.** For each $k=1,2,\cdots,l$ , we have $\wedge\mathbf{R}_k\leq\vee c_{ij}$ . By the disjunction and conjunction laws, $\mathbf{R}_k\cap c_{ij}\neq\varnothing$ for any $c_{ij}\neq\varnothing$ . Since $g_D(U,\mathbf{R})=(\wedge\mathbf{R}_1)\vee(\wedge\mathbf{R}_2)\vee\cdots\vee(\wedge\mathbf{R}_l)$ , it follows that for arbitrary $\mathbf{R}_k$ if we reduce an element $R$ from $\mathbf{R}_k$ , let $\mathbf{R}_k'=\mathbf{R}_k-\{R\}$ , then $g_D(U,\mathbf{R})\neq\overset{k-1}{\underset{r=1}{\vee}}(\wedge\mathbf{R}_r)\vee(\wedge\mathbf{R}_k')\vee\overset{l}{\underset{r=k+1}{\vee}}(\wedge\mathbf{R}_r)$ and $g_D(U,\mathbf{R})<\overset{k-1}{\underset{r=1}{\vee}}(\wedge\mathbf{R}_r)\vee(\wedge\mathbf{R}_k')\vee\overset{l}{\underset{r=k+1}{\vee}}(\wedge\mathbf{R}_r)$ . If we still have $\mathbf{R}_k' \cap c_{ij} \neq \varnothing$ for any $c_{ij} \neq \varnothing$, then $\wedge \mathbf{R}_k' \leq \vee c_{ij}$ for any $c_{ij} \neq \varnothing$. This implies that $g_D(U,\mathbf{R}) \geq \mathop{\vee}\limits_{r=1}^{k-1}(\wedge \mathbf{R}_r) \vee (\wedge \mathbf{R}_k') \vee \mathop{\vee}\limits_{r=k+1}^{l}(\wedge \mathbf{R}_r)$ and $g_D(U,\mathbf{R}) = \mathop{\vee}\limits_{r=1}^{k-1}(\wedge \mathbf{R}_r) \vee (\wedge \mathbf{R}_k') \vee \mathop{\vee}\limits_{r=k+1}^{l}(\wedge \mathbf{R}_r)$, which is a contradiction. Hence, there exists $g_D(U,\mathbf{R}) \neq \mathop{\vee}\limits_{r=1}^{k-1}(\wedge \mathbf{R}_r) \vee (\wedge \mathbf{R}_k') \vee \mathop{\vee}\limits_{r=k+1}^{l}(\wedge \mathbf{R}_r)$ and $c_{i_0 j_0} \neq \varnothing$ such that $\mathbf{R}_k' \cap c_{i_0 j_0} = \varnothing$, which implies that $\mathbf{R}_k$ is a relative reduction of $\mathbf{R}$.

For any $\mathbf{X} \in Red_D(\mathbf{R})$, we have $\mathbf{X} \cap c_{ij} \neq \varnothing$ for any $c_{ij} \neq \varnothing$. Thus $f_D(U,\mathbf{R}) \wedge (\wedge \mathbf{X}) = \wedge(\vee c_{ij}) \wedge (\wedge \mathbf{X}) = (\wedge \mathbf{X})$. This implies $\wedge \mathbf{X} \leq f_D(U,\mathbf{R}) = g_D(U,\mathbf{R})$. If $\mathbf{R}_k - \mathbf{X} \neq \varnothing$ for each $k$, we can find $R_k \in \mathbf{R}_k - \mathbf{X}$ for each $k$. By rewriting $g_D(U,\mathbf{R}) = \left(\mathop{\vee}\limits_{k=1}^{l} R_k\right) \wedge \varnothing$, we have $\wedge \mathbf{X} \leq \mathop{\vee}\limits_{k=1}^{l} R_k$. So there must be $R_{k_0}$ such that $\wedge \mathbf{X} \leq R_{k_0}$, this implies $R_{k_0} \in \mathbf{X}$. This is a contradiction. So $\mathbf{R}_{k_0} \in \mathbf{X}$ for some $k_0$. Since both $\mathbf{X}$ and $\mathbf{R}_{k_0}$ are relative reductions, we have $\mathbf{X} = \mathbf{R}_{k_0}$. Hence $Red_D(\mathbf{R}) = \{\mathbf{R}_1, \cdots, \mathbf{R}_l\}$.

**Remark 3.1.** If the interval-valued fuzzy similarity relation is a fuzzy similarity relation, then the lower approximation degenerates into the fuzzy one [43], and our method in this section coincides with the fuzzy one found in [23]. If the interval-valued fuzzy similarity relation is a crisp equivalence relation, then the lower approximation degenerates into the crisp one [1], and our method in this section coincides with the crisp one found in [14]. Thus, our idea and method are really the generalization of the fuzzy one found in [23] and the crisp one found in [14] for interval-valued fuzzy case.

It should be noted that if $c_{ij} \cap Core_D(\mathbf{R}) \neq \varnothing$, then $\{R\} \wedge (\vee c_{ij}) = \{R\}$ for $R \in c_{ij} \cap Core_D(\mathbf{R})$. When computing $g_D(U,\mathbf{R})$ by $f_D(U,\mathbf{R})$ we can only consider elements in $Core_D(\mathbf{R})$ and $c_{ij}$ satisfying $c_{ij} \cap Core_D(\mathbf{R}) = \varnothing$ so that the computational load may be reduced. We can design an algorithm to compute reductions for interval-valued fuzzy decision systems.

Suppose $U = \{x_1, x_2, \cdots, x_n\}$, $U/D = \{D_1, D_2, \cdots, D_s\}$.

Step 1: Compute $Sim(\mathbf{R})$.

Step 2: Compute $\underline{Sim(\mathbf{R})}(D_k)$ for every $D_k \in U/D$.

Step 3: Compute $c_{ij}$: if $\lambda_{j1} < \lambda_{i1}$ and $\lambda_{j2} < \lambda_{i2}$, then $c_{ij} = \{R : 1 - R^+(x_i, x_j) \geq \lambda_{i1} \ or \ 1 - R^-(x_i, x_j) \geq \lambda_{i2}\}$, otherwise $c_{ij} = \varnothing$.

Step 4: Compute core as collection of those $c_{ij}$ with single element.

Step 5: Delete those $c_{ij} = \varnothing$ and $c_{ij}$ with nonempty overlap with the core.

Step 6: Define $f_D(U,\mathbf{R}) = \wedge\{\vee(c_{ij})\}$ with $c_{ij}$ left after Step 5.

Step 7: Compute $g_D(U,\mathbf{R}) = (\wedge\mathbf{R}_1) \vee (\wedge\mathbf{R}_2) \vee \cdots \vee (\wedge\mathbf{R}_l)$ by $f_D(U,\mathbf{R}) = \wedge\{\vee(c_{ij})\}$
.

Step 8: Output all reductions $Red_D(\mathbf{R}) = \{\mathbf{R}_1, \cdots, \mathbf{R}_l\}$ .

# 4 An Illustrated Example

The following example of data set (adopted from the [44,45] with some modifications) is employed to illustrate our idea in Section 3.

**Example 4.1.** The car data set contains the information of ten new cars. Let $U = \{x_1, x_2, \cdots, x_{10}\}$ be the cars, each of which is described by six attributes: (1) $C_1$: fuel economy; (2) $C_2$: aerod degree; (3) $C_3$: price; (4) $C_4$: comfort; (5) $C_5$: design; and (6) $C_6$: safety. The characteristics of the ten new cars under the six attributes are represented by the interval-valued fuzzy sets, as shown in Table 1.

Every IVF attribute $C_k$ can define an IVF similarity relation $R_k$ as

$$R_k(x_i, x_j) = \begin{cases} \min\{C_k(x_i), C_k(x_j)\}, & C_k(x_i) \neq C_k(x_j) \\ \bar{1}, & C_k(x_i) = C_k(x_j) \end{cases}$$

**Table 1. The car data set**

| $U$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | [0.3,0.4] | [0.5,0.6] | [0.6,0.7] | [0.7,0.8] | [0.6,0.7] | [0.2,0.7] |
| $x_2$ | [0.6,0.8] | [0.6,0.7] | [0.2,0.3] | [0.5,0.6] | [0.7,0.8] | [0.3,0.9] |
| $x_3$ | [0.5,0.8] | [0.7,0.8] | [0.5,0.5] | [0.2,0.3] | [0.4,0.6] | [0.5,0.8] |
| $x_4$ | [0.2,0.3] | [0.5,0.7] | [0.6,0.7] | [0.4,0.5] | [0.6,0.9] | [0.4, 0.7] |
| $x_5$ | [0.6,0.8] | [0.3,0.5] | [0.4,0.6] | [0.6,0.8] | [0.5,0.6] | [0.6,0.6] |
| $x_6$ | [0.4,0.6] | [0.3,0.5] | [0.3,0.6] | [0.7,0.8] | [0.4,0.5] | [0.2,0.7] |
| $x_7$ | [0.6,0.7] | [0.4,0.5] | [0.7,0.8] | [0.3,0.6] | [0.3,0.7] | [0.1,0.6] |
| $x_8$ | [0.7,0.7] | [0.3,0.9] | [0.1,0.2] | [0.4,0.5] | [0.4,0.5] | [0.8,0.9] |
| $x_9$ | [0.4,0.5] | [1,1] | [0.4,0.5] | [0.6,0.7] | [0.2,0.7] | [0.1,0.8] |
| $x_{10}$ | [0.7,0.8] | [0.5,0.9] | [0.3,0.4] | [0.5,0.9] | [0.5,0.7] | [0.4,0.6] |

$Sim(\mathbf{R})$ can be computed as

$$Sim(\mathbf{R})(x_i,x_j)=\begin{pmatrix} \bar{1} & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.2,0.4] & [0.3,0.4] & [0.1,0.4] & [0.1,0.2] & [0.1,0.4] & [0.2,0.4] \\ [0.2,0.3] & \bar{1} & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.1,0.3] & [0.1,0.2] & [0.1,0.3] & [0.2,0.3] \\ [0.2,0.3] & [0.2,0.3] & \bar{1} & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.1,0.3] & [0.1,0.2] & [0.1,0.3] & [0.2,0.3] \\ [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & \bar{1} & [0.2,0.3] & [0.2,0.3] & [0.1,0.3] & [0.1,0.2] & [0.1,0.3] & [0.2,0.3] \\ [0.2,0.4] & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & \bar{1} & [0.2,0.5] & [0.1,0.5] & [0.1,0.2] & [0.1,0.5] & [0.3,0.4] \\ [0.3,0.4] & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.2,0.5] & \bar{1} & [0.1,0.5] & [0.1,0.2] & [0.1,0.5] & [0.2,0.4] \\ [0.1,0.4] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.5] & [0.1,0.5] & \bar{1} & [0.1,0.2] & [0.1,0.5] & [0.1,0.4] \\ [0.1,0.2] & [0.1,0.2] & [0.1,0.2] & [0.1,0.2] & [0.1,0.2] & [0.1,0.2] & [0.1,0.2] & \bar{1} & [0.1,0.2] & [0.1,0.2] \\ [0.1,0.4] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.5] & [0.1,0.5] & [0.1,0.5] & [0.1,0.2] & \bar{1} & [0.1,0.4] \\ [0.2,0.4] & [0.2,0.3] & [0.2,0.3] & [0.2,0.3] & [0.3,0.4] & [0.2,0.5] & [0.1,0.4] & [0.1,0.2] & [0.1,0.4] & \bar{1} \end{pmatrix}$$

Suppose a decision partition is $A=\{x_1,x_2,x_4,x_7,x_9\}$, $B=\{x_3,x_5,x_6,x_8,x_{10}\}$ then

$$\underline{Sim(\mathbf{R})}(A)(x)=\begin{cases}[0.6,0.7], x=x_1\\ [0.7,0.8], x=x_2\\ [0.7,0.8], x=x_4\\ [0.5,0.9], x=x_7\\ [0.5,0.9], x=x_9\\ [0,0], \quad \text{other}\end{cases} \qquad \underline{Sim(\mathbf{R})}(B)(x)=\begin{cases}[0.7,0.8], x=x_3\\ [0.5,0.8], x=x_5\\ [0.5,0.7], x=x_6\\ [0.8,0.9], x=x_8\\ [0.6,0.8], x=x_{10}\\ [0,0], \quad \text{other}\end{cases}$$

and the discernibility matrix $M_D(U,\mathbf{R})$ of $(c_{ij})$ is as follows:

$$\begin{pmatrix} \varnothing & \varnothing & \{1,4,6\} & \varnothing & \{1,2,6\} & \{1,2,3\} & \varnothing & \{1,2,3,6\} & \varnothing & \{1,3,6\} \\ \{3,6\} & \varnothing & \{3,4\} & \varnothing & \{3\} & \{3,6\} & \varnothing & \{3\} & \varnothing & \{3\} \\ \{4,6\} & \{3,4\} & \varnothing & \{1,4\} & \varnothing & \{4,6\} & \{4,6\} & \varnothing & \{4,5,6\} & \varnothing \\ \{1,6\} & \varnothing & \{1,4\} & \varnothing & \{1\} & \{1,6\} & \varnothing & \{1,3\} & \varnothing & \{1\} \\ \{1,2,6\} & \{2,3\} & \varnothing & \{1,2,4\} & \varnothing & \varnothing & \{2,6\} & \varnothing & \{1,2,3,5,6\} & \varnothing \\ \{1,2,3,5\} & \{2,3,5,6\} & \varnothing & \{1,2,3,4,5,6\} & \varnothing & \varnothing & \{2,3,4,5,6\} & \varnothing & \{1,2,3,5,6\} & \varnothing \\ \varnothing & \varnothing & \{2,3,4,6\} & \varnothing & \{2,6\} & \{2,5,6\} & \varnothing & \{2,3,4,5,6\} & \varnothing & \{2,3,6\} \\ \{3\} & \{3\} & \{3\} & \{3\} & \{3\} & \{3\} & \{3,6\} & \varnothing & \{3,6\} & \{3\} \\ \varnothing & \varnothing & \{1,3,4,6\} & \varnothing & \{1,2,3,6\} & \{1,2,3,5,6\} & \varnothing & \{1,3,4,5,6\} & \varnothing & \{1,3,6\} \\ \{1,3,6\} & \{3\} & \varnothing & \{1,3\} & \varnothing & \{3,6\} & \{3,6\} & \varnothing & \{3,5,6\} & \varnothing \end{pmatrix}$$

We can get that $Core_D(\mathbf{R})=\{C_1,C_3\}$ and

$$Red_D(\mathbf{R})=\{\{C_1,C_2,C_3,C_4\},\{C_1,C_2,C_3,C_6\},\{C_1,C_3,C_6\}\}.$$

Based on the above analysis, we can see that the proposed Algorithm has a less computational complexity.

# 5 Conclusion

The aim of this paper is to focus on attributes reduction based on interval-valued fuzzy rough sets. After reviewing attributes reduction with traditional rough sets, some equivalent conditions to describe the relative reduction based on interval-valued fuzzy rough sets are proposed, and the structure of reduction is completely examined. An algorithm based on discernibility matrix to compute all the attributes reductions has been developed. At last the concepts of attributes reduction have been demonstrated by an example. This work may be viewed as the extension of [14] and [23] in the interval-valued fuzzy environment. In the future, our work will focus on the two facets. On one hand, we will study computational complexity of the proposed algorithm in this paper. On the other hand, we will concentrate our discussion on some fast algorithms to compute attributes reduction.


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# Authors' Contributions

'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol and wrote the overall draft of the manuscript. The author read and approved the final manuscript.

# Competing Interests

Author has declared that no competing interests exist.

# References


[1] Pawlak Z. Rough Set. International Journal of Computer and Information Science. 1982;11:341-356.

[2] Skowron A, Polkowski L. Rough sets in knowledge discovery. Springer, Berlin. 1998;1-2.

[3] Slowinski R. Intelligent decision support: Handbook of applications and advances of the rough sets theory. Kluwer Academic Publishers, Boston; 1992.

[4] Ziarko WP. Rough sets, fuzzy sets and knowledge discovery. Workshop in Computing, Springer, London; 1994.

[5] Jagielska I, Matthews C, Whitfort T. An investigation into the application of neural networks, fuzzy logic, genetic algorithms, and rough sets to automated knowledge acquisition for classification problems. Neurocomputing. 1999;24(1-3):37-54.

[6] Kryszkiewicz M. Rough set approach to incomplete information systems. Information Sciences. 1998;112(1-4):39-49.

[7] Pawlak Z. Rough sets. Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers, Boston; 1991.

[8] Tsumoto S. Automated extraction of medical expert system rules from clinical databases based on rough set theory. Information Sciences. 1998;112(1-4):67-84.

[9] Hu QH, Xie ZX, Yu DR. Hybrid Attribute Reduction Based on a Novel Fuzzy Rough Model and Information Granulation. Pattern Recognition. 2007;40(12):3509-3521.

[10] Hu QH, Yu DR, Xie ZX. Information-Preserving Hybrid Data Reduction Based on Fuzzy-Rough Techniques. Pattern Recognition Letters. 2006;27(5):414-423.

[11] Jensen R, Shen Q. Fuzzy-rough attributes reduction with application to web categorization. Fuzzy Sets and Systems. 2004;141:469-485.

[12] Jensen R, Shen Q. Fuzzy-rough Sets Assisted Attribute Selection. IEEE Transactions on Fuzzy Systems. 2007;15(1):73-89.

[13] Leung Y, Wu WZ, Zhang WX. Knowledge Acquisition in Incomplete Information Systems: A Rough Set Approach. European Journal of Operational Research. 2006;168(1):164-180.

[14] Skowron A, Rauszer C. The discernibility matrices and functions in information systems, in Intelligent Decision Support, Handbook of Applications and Advances of the Rough Sets Theory. Slowinski R, Ed. Norwell, MA: Kluwer; 1992.

[15] Wu WZ, Li HZ, Zhang WX. Knowledge Acquisition in Incomplete Fuzzy Information Systems via Rough Set Approach. Expert Systems. 2003;20(5):280-285.

[16] Yao YY. Relational Interpretations of Neighborhood Operators and Rough Set Approximation Operators. Information Sciences. 1998;111(1-4):239-259.

[17] Yao YY, Zhao Y. Attribute reduction in decision-theoretic rough set models. Information Sciences. 2008;178(17):3356-3373.

[18] Zhao SY, Tsang ECC. On fuzzy approximation operators in attribute reduction with fuzzy rough sets. Information Sciences. 2008;178(16):3163-3176.

[19] Zhu W, Wang FY. Reduction and Axiomization of Covering Generalized Rough Sets. Information Sciences. 2003;152:217-230.

[20] Zhu W, Wang FY. On Three Types of Covering-Based Rough Sets. IEEE Transactions on Knowledge and Data Engineering. 2007;19(8):1131-1144.

[21] Chen DG, Wang CZ, Hu QH. A New Approach to Attribute Reduction of Consistent and Inconsistent Covering Decision Systems with Covering Rough Sets. Information Sciences. 2007;177(17):3500-3518.

[22] Tsang ECC, Chen DG, Yeung DS. Approximations and reducts with covering generalized rough sets. Computers & Mathematics with Applications. 2008;56(1):279-289.

[23] Tsang ECC, Chen DG, Yeung DS, Wang XZ, Lee JWT. Attributes reduction using fuzzy rough sets. IEEE Transactions on Fuzzy Systems. 2008;16(5):1130-1141.

[24] Wang CZ, Wu CX, Chen DG. A systematic study on attribute reduction with rough sets based on general binary relations. Information Sciences. 2008;178(9):2237-2261.

[25] Turksen LB. Interval valued fuzzy sets based on normal forms. Fuzzy Sets and Systems. 1986;20(2):191-210.

[26] Zadeh LA. The concepts of linguistic variable and its application to approximate reasoning, part I. Information Sciences. 1975;8(3):199-249.

[27] Zadeh LA. Fuzzy Sets. Information and Control. 1965;8:338-353.

[28] Bustince H. Indicator of inclusion grade for interval-valued fuzzy sets, application to approximate reasoning based on interval-valued fuzzy sets. International Journal of Approximate Reasoning. 2000;23(3):137-209.

[29] Bustince H, Barrenechea E, Pagola M, Fernandez J. Interval-valued fuzzy sets constructed from matrices: Application to edge detection. Fuzzy Sets and Systems. 2009;160(13):1819-1840.

[30] Chen SM, Wang HY. Evaluating students' answerscripts based on interval-valued fuzzy grade sheets. Expert Systems with Applications. 2009;36(6):9839-9846.

[31] Gorzalczany MB. A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets and Systems. 1987;21(1):1-17.

[32] Gorzalczany MB. An interval-valued fuzzy inference method-Some basic properties. Fuzzy Sets and Systems. 1989;31(2):243-251.

[33] Guh YY, Yang MS, Po RW, Lee ES. Interval-valued fuzzy relation-based clustering with its application to performance evaluation. Computers & Mathematics with Applications. 2009;57(5):841-849.

[34] Lu HW, Huang GH, He L. Development of an interval-valued fuzzy linear-programming method based on infinite α-cuts for water resources management. Environmental Modelling & Software. 2010;25(3):354-361.

[35] Miao DQ, Zhao Y, Yao YY, Li HX, Xu FF. Relative reducts in consistent and inconsistent decision tables of the Pawlak rough set model. Information Sciences. 2009;179(24):4140-4150.

[36] Sambuc R. Functions Φ -Flous Application al' aide au Diagnostic en Pathologie Thyroidienne. These of Doctorat in Merseille ; 1975.

[37] Turksen IB, Zhong Z. An approximate analogical reasoning schema based on similarity measures and interval-valued fuzzy sets. Fuzzy Sets and Systems. 1990;34(3):323-346.

[38] Wei SH, Chen SM. Fuzzy risk analysis based on interval-valued fuzzy numbers. Expert Systems with Applications. 2009;36(2):2285-2299.

[39] Yager RR. Level sets and the extension principle for interval valued fuzzy sets and its application to uncertainty measures. Information Sciences. 2008;178(18):3565-3576.

[40] Gong ZT, Sun BZ, Chen DG. Rough Set Theory for Interval-Valued Fuzzy Information Systems. Information Sciences. 2008;178(8):1968-1985.

[41] Sun BZ, Gong ZT, Chen DG. Fuzzy Rough Set Theory for the Interval-Valued Fuzzy Information Systems. Information Sciences. 2008;178(13):2794-2815.

[42] Zhang HY, Zhang WX, Wu WZ. On characterization of generalized interval-valued fuzzy rough sets on two universes of discourse. International Journal of Approximate Reasoning. 2009;51(1):56-70.

[43] Dubois D, Prade H. Rough Fuzzy Sets and Fuzzy Rough Sets. International Journal of General Systems. 1990;17(2):191-209.

[44] Herrera F, Martinez L. An approach for combining linguistic and numerical information based on2-tuple fuzzy linguistic representation model in decision-making. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 2008;5:539-562.

[45] Xu ZS. Intuitionistic Fuzzy Information Integration Theory and Applications. Science Press, Beijing; 2008.

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