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The Convolution Sums $\sum_{m=1}^{n-1} m^k \sigma_e(m)\sigma_f(n-m)$

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Abstract

In this paper we obtain some various convolution sum formulae especially, focusing on the form

$$\sum_{m=1}^{n-1} m^k \sigma_e(m)\sigma_f(n-m)$$

for $k \in \mathbb{N}$ $(1 \le k \le 5)$ and an odd positive integer e and f. Furthermore, we obtain some identities related to the convolution sums.

Keywords: Divisor functions; Convolution sums

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1 Introduction

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define necessary divisor function and $\Delta(q)$ which appear in many areas of number theory:

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad \Delta(q) := \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{1.1}$$

The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are given by

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.2}$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.3}$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \tag{1.4}$$

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in ([1], p. 318). Lahiri ([2], p. 149) has derived the following identities from the work of Ramanujan [3] :

$$L^2(q) = 1 - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{1.5}$$

$$L^3(q) = 1 - 1728\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 2160\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.6}$$

$$\begin{aligned} L^4(q) = 1 &- 6912\sum_{n=1}^{\infty} n^3\sigma_1(n)q^n + 10368\sum_{n=1}^{\infty} n^2\sigma_3(n)q^n - 4032\sum_{n=1}^{\infty} n\sigma_5(n)q^n \\ &+ 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \end{aligned} \tag{1.7}$$

$$\begin{aligned} L^5(q) = 1 &- 20736\sum_{n=1}^{\infty} n^4\sigma_1(n)q^n + 34560\sum_{n=1}^{\infty} n^3\sigma_3(n)q^n - 17280\sum_{n=1}^{\infty} n^2\sigma_5(n)q^n \\ &+ 3600\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n, \end{aligned} \tag{1.8}$$

$$M^2(q) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.9}$$

$$M^3(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{432000}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \tag{1.10}$$

$$L(q)M(q) = 1 + 720\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.11}$$

$$L^2(q)M(q) = 1 + 1728\sum_{n=1}^{\infty} n^2\sigma_3(n)q^n - 2016\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.12}$$

$$L(q)M^2(q) = 1 + 720\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n, \tag{1.13}$$

$$L(q)N(q) = 1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.14}$$

$$M(q)N(q) = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n, \tag{1.15}$$

Here for $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we define

$$\begin{aligned} I^k_{m,e,f}(n) &:= \sum_{m=1}^{n-1} m^k\sigma_e(m)\sigma_f(n-m), \\ I_{m,e,f}(n) := I^1_{m,e,f}(n) &= \sum_{m=1}^{n-1} m\sigma_e(m)\sigma_f(n-m), \\ I_{e,f}(n) := I^0_{m,e,f}(n) &= \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m). \end{aligned} \tag{1.16}$$

Ramanujan [3] and Lahiri [2], [4] have shown that $I_{e,f}$ can be expressed as :

$$
\begin{aligned}
I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n),\\
I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),\\
I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),\\
I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n),\\
I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n),\\
I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n),\\
I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n),\\
I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n),\\
I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n),\\
I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n),\\
I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n),\\
I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
\end{aligned}
$$

Also we have already obtained :

Proposition 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) *(See ([5], p. 155))*

$$I_{m,1,1}(n) = \frac{1}{24}n\left\{5\sigma_3(n) - (6n-1)\,\sigma_1(n)\right\},$$

(b) *(See ([5], p. 157))*

$$
\begin{aligned}
I_{m,1,3}(n) &= \frac{7}{240}n\sigma_5(n) - \frac{1}{40}n^2\sigma_3(n) - \frac{1}{240}n\sigma_1(n),\\
I_{m,3,1}(n) &= \frac{7}{120}n\sigma_5(n) + \left(\frac{1}{24}n - \frac{1}{10}n^2\right)\sigma_3(n),
\end{aligned}
$$

(c) *(See ([6], Theorem 3.3, Theorem 3.1), ([5], p. 155))*

$$
\begin{aligned}
I_{m,1,5}(n) &= \frac{1}{504}n\left\{5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n)\right\},\\
I_{m,3,3}(n) &= \frac{1}{240}n\left\{\sigma_7(n) - \sigma_3(n)\right\},\\
I_{m,5,1}(n) &= \frac{1}{168}n\left\{5\sigma_7(n) - (12n-7)\,\sigma_5(n)\right\},
\end{aligned}
$$

(d) *(See ([6], Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5))*

$$I_{m,1,7}(n) = \frac{1}{7200}\left\{33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n)\right\},$$
$$I_{m,3,5}(n) = \frac{1}{12600}\left\{11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n)\right\},$$
$$I_{m,5,3}(n) = \frac{1}{8400}\left\{11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n)\right\},$$
$$I_{m,7,1}(n) = \frac{1}{1800}\left\{33n\sigma_9(n) - 25n\,(4n-3)\,\sigma_7(n) - 8\tau(n)\right\}.$$

In this paper we mainly focus on obtaining the convolution sum formulae as :

Theorem 1.1. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$I^2_{m,1,5}(n) = \sum_{m=1}^{n-1} m^2\sigma_1(m)\sigma_5(n-m)$$
$$= \frac{1}{3024}\left\{10n^2\sigma_7(n) - 9n^3\sigma_5(n) + 6n^2\sigma_1(n) - 7\tau(n)\right\},$$

(b)

$$I^2_{m,3,3}(n) = \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_3(n-m) = \frac{1}{2160}\left\{5n^2\sigma_7(n) - 9n^2\sigma_3(n) + 4\tau(n)\right\},$$

(c)

$$I^2_{m,5,1}(n) = \sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_1(n-m)$$
$$= \frac{1}{432}\left\{10n^2\sigma_7(n) - 9n^2\,(3n-2)\,\sigma_5(n) - \tau(n)\right\},$$

(d)

$$I^2_{m,1,7}(n) = \sum_{m=1}^{n-1} m^2\sigma_1(m)\sigma_7(n-m)$$
$$= \frac{1}{7200}n\left\{9n\sigma_9(n) - 10n^2\sigma_7(n) - 15n\sigma_1(n) + 16\tau(n)\right\},$$

(e)

$$I^2_{m,3,5}(n) = \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_5(n-m) = \frac{1}{2520}n\left\{n\sigma_9(n) + 5n\sigma_3(n) - 6\tau(n)\right\},$$

(f)

$$I^2_{m,5,3}(n) = \sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_3(n-m) = \frac{1}{1200}n\left\{n\sigma_9(n) - 5n\sigma_5(n) + 4\tau(n)\right\},$$

(g)

$$I^2_{m,7,1}(n) = \sum_{m=1}^{n-1} m^2\sigma_7(m)\sigma_1(n-m)$$
$$= \frac{1}{600} n \left\{9n\sigma_9(n) - 5n\,(6n-5)\,\sigma_7(n) - 4\tau(n)\right\}.$$

Moreover we calculate

$$I^3_{m,5,1}(n) = \sum_{m=1}^{n-1} m^3\sigma_5(m)\sigma_1(n-m)$$
$$= \frac{1}{216}\left\{4n^3\sigma_7(n) - 3n^3\,(4n-3)\,\sigma_5(n) - n\tau(n)\right\},$$
$$I^4_{m,3,1}(n) = \sum_{m=1}^{n-1} m^4\sigma_3(m)\sigma_1(n-m)$$
$$= \frac{1}{288}\left\{7n^4\sigma_5(n) - 6n^4\,(3n-2)\,\sigma_3(n) - n\tau(n)\right\},$$
$$I^5_{m,1,1}(n) = \sum_{m=1}^{n-1} m^5\sigma_1(m)\sigma_1(n-m)$$
$$= \frac{1}{336}\left\{15n^5\sigma_3(n) - 14n^5\,(2n-1)\,\sigma_1(n) - n\tau(n)\right\}$$

(see Theorem 2.6 (e), Theorem 2.8 (c), and Theorem 2.10). And in Section 2, we obtain some identities related to the convolution sums, for example,

Lemma 1.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$L^7(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n + \frac{458640}{691}\sum_{n=1}^{\infty} n\sigma_{11}(n)q^n - \frac{36288}{5}\sum_{n=1}^{\infty} n^2\sigma_9(n)q^n$$
$$+ 40320\sum_{n=1}^{\infty} n^3\sigma_7(n)q^n - 120960\sum_{n=1}^{\infty} n^4\sigma_5(n)q^n + 186624\sum_{n=1}^{\infty} n^5\sigma_3(n)q^n$$
$$- \frac{497664}{5}\sum_{n=1}^{\infty} n^6\sigma_1(n)q^n - \frac{4608}{3455}\sum_{n=1}^{\infty} n\tau(n)q^n.$$

Finally, by using the above results we can easily obtain the following theorem :

Theorem 1.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_1(a)\sigma_1(b)\sigma_1(c) = \sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_1(k)\sigma_1(m-k)\sigma_1(n-m)$$
$$= \frac{1}{576} n\left\{7\sigma_5(n) - 10\,(3n-1)\,\sigma_3(n) + \left(24n^2 - 12n + 1\right)\sigma_1(n)\right\},$$

(b)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_3(b)\sigma_1(c) = \frac{1}{5760} n \left\{5\sigma_7(n) - (18n-7)\,\sigma_5(n) + \left(12n^2 - 6n - 5\right)\sigma_3(n) + (6n-1)\,\sigma_1(n)\right\},$$

(c)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_3(a)\sigma_1(b)\sigma_1(c) = \frac{1}{2880} n \left\{5\sigma_7(n) - 2\,(12n-7)\,\sigma_5(n) + \left(24n^2 - 24n + 5\right)\sigma_3(n)\right\},$$

(d)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_5(b)\sigma_1(c) = \frac{1}{60480} \left\{11n\sigma_9(n) - 25n\,(2n-1)\,\sigma_7(n) + 15n^2\,(3n-2)\,\sigma_5(n) + 25n\sigma_3(n) - 5n\,(6n-1)\,\sigma_1(n) - \tau(n)\right\},$$

(e)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_3(a)\sigma_3(b)\sigma_1(c) = \frac{1}{432000} \left\{33n\sigma_9(n) - 25n\,(4n-3)\,\sigma_7(n) - 105n\sigma_5(n) + 15n\,(12n-5)\,\sigma_3(n) - 8\tau(n)\right\},$$

(f)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_5(a)\sigma_1(b)\sigma_1(c) = \frac{1}{60480} \left\{33n\sigma_9(n) - 50n\,(4n-3)\,\sigma_7(n) + 15n\left(18n^2 - 24n + 7\right)\sigma_5(n) + 2\tau(n)\right\},$$

(g)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_1(a)\sigma_7(b)\sigma_1(c) = \frac{1}{119404800} \left\{6825n\sigma_{11}(n) - 2073n\,(18n-11)\,\sigma_9(n) + 6910n^2\,(6n-5)\,\sigma_7(n) - 51825n\sigma_3(n) + 10365n\,(6n-1)\,\sigma_1(n) - 8\,(2667n-2764)\,\tau(n)\right\},$$

(h)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} a\sigma_3(a)\sigma_5(b)\sigma_1(c) = \frac{1}{208958400} \left\{2275n\sigma_{11}(n) - 691n\,(12n-11)\,\sigma_9(n) + 24185n\sigma_5(n) - 3455n\,(12n-5)\,\sigma_3(n) + 36\,(647n-691)\,\tau(n)\right\},$$

(i)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_5(a)\sigma_3(b)\sigma_1(c) = \frac{1}{139305600}\left\{2275n\sigma_{11}(n) - 691n\,(12n-11)\,\sigma_9(n)\right.$$
$$- 17275n\sigma_7(n) + 3455n\,(12n-7)\,\sigma_5(n)$$
$$\left.-24\,(757n-691)\,\tau(n)\right\},$$

(j)

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_7(a)\sigma_1(b)\sigma_1(c) = \frac{1}{29851200}\left\{6825n\sigma_{11}(n) - 4146n\,(12n-11)\,\sigma_9(n)\right.$$
$$+ 3455n\left(24n^2-40n+15\right)\sigma_7(n)$$
$$\left.+8\,(1479n-1382)\,\tau(n)\right\}.$$

2 The various convolution sums multiplying m^k for a positive integer k

We obtain the similar results as Proposition 1.1 in the following theorem and in Theorem 2.3 :

Theorem 2.1. *Let* $n\in\mathbb{N}$*. Then we have*

(a)

$$I_{m,3,7}(n) = \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_7(n-m)$$
$$\frac{1}{331680}n\left\{91\sigma_{11}(n) - 691\sigma_3(n) + 600\tau(n)\right\},$$

(b)

$$I_{m,5,5}(n) = \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m)$$
$$= \frac{1}{348264}n\left\{65\sigma_{11}(n) + 691\sigma_5(n) - 756\tau(n)\right\},$$

(c)

$$I_{m,7,3}(n) = \sum_{m=1}^{n-1} m\sigma_7(m)\sigma_3(n-m)$$
$$= \frac{1}{165840}n\left\{91\sigma_{11}(n) - 691\sigma_7(n) + 600\tau(n)\right\}.$$

Proof. (a) By (1.9) and (1.11) we have

$$\begin{aligned}
&L(q)M^3(q) = M^2(q)\cdot L(q)M(q)\\
&= \left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\left(1+720\sum_{m=1}^{\infty}m\sigma_3(m)q^m - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{720N\sigma_3(N)-504\sigma_5(N)+480\sigma_7(N)\\
&\quad +480\cdot 720\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m\sigma_3(m) - 480\cdot 504\sum_{m=1}^{N-1}\sigma_7(N-m)\sigma_5(m)\Bigg\}q^N \\
&= 1+\sum_{N=1}^{\infty}\{720N\sigma_3(N)-504\sigma_5(N)+480\sigma_7(N)+480\cdot 720\cdot I_{m,3,7}(N)\\
&\quad -480\cdot 504\cdot I_{5,7}(N)\}\,q^N,
\end{aligned} \tag{2.1}$$

where we use the definition of $I^k_{m,e,f}(n)$ in (1.16). Also, by (1.3) and (1.13), we obtain

$$\begin{aligned}
&L(q)M^3(q) = L(q)M^2(q)\cdot M(q)\\
&= \left(1+720\sum_{n=1}^{\infty}n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n\right)\left(1+240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{720N\sigma_7(N)-264\sigma_9(N)+240\sigma_3(N)\\
&\quad + 720\cdot 240\sum_{m=1}^{N-1}(N-m)\,\sigma_7(N-m)\sigma_3(m)\\
&\quad -264\cdot 240\sum_{m=1}^{N-1}\sigma_9(N-m)\sigma_3(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\{720N\sigma_7(N)-264\sigma_9(N)+240\sigma_3(N)+720\cdot 240N\cdot I_{3,7}(N)\\
&\quad -720\cdot 240\cdot I_{m,3,7}(N)-264\cdot 240\cdot I_{3,9}(N)\}\,q^N.
\end{aligned} \tag{2.2}$$

So equating (2.1) with (2.2) we can complete the proof.

(b) We observe that

$$I_{m,5,5}(n) = \sum_{m=1}^{n-1}m\sigma_5(m)\sigma_5(n-m) = \sum_{m=1}^{n-1}(n-m)\,\sigma_5(n-m)\sigma_5(m),$$

which leads that

$$I_{m,5,5}(n) = \frac{n}{2}\cdot I_{5,5}(n).$$

(c) We note that

$$\begin{aligned}
I_{m,7,3}(n) = \sum_{m=1}^{n-1}m\sigma_7(m)\sigma_3(n-m) &= \sum_{m=1}^{n-1}(n-m)\,\sigma_7(n-m)\sigma_3(m)\\
&= n\cdot I_{3,7}(n) - I_{m,3,7}(n).
\end{aligned}$$

Thus we use Theorem 2.1 (a).

□

Theorem 2.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$L(q)M^3(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}n\sigma_{11}(n)q^n + \frac{432000}{691}\sum_{n=1}^{\infty}n\tau(n)q^n.$$

Proof. Insert Theorem 2.1 (a) into (2.1).

□

And we can see the following proposition :

Proposition 2.1. *(See ([6], Lemma 3.6, Lemma 3.4)) For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$\begin{aligned} L^2(q)M^2(q) = 1 + 960\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n - \frac{3168}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n \\ + \frac{37248}{3455}\sum_{n=1}^{\infty}\tau(n)q^n, \end{aligned}$$

(b)

$$\begin{aligned} L(q)M(q)N(q) = 1 - \frac{1584}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n \\ - \frac{228096}{3455}\sum_{n=1}^{\infty}\tau(n)q^n. \end{aligned}$$

Theorem 2.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n-1}\sigma_1(m)\tau(n-m) = -\frac{1}{24}(n-1)\tau(n),$$

(b)

$$\begin{aligned} I_{m,1,9}(n) &= \sum_{m=1}^{n-1}m\sigma_1(m)\sigma_9(n-m) \\ &= \frac{1}{912120}n\left\{2275\sigma_{11}(n) - 4146n\sigma_9(n) + 3455\sigma_1(n) - 1584\tau(n)\right\}, \end{aligned}$$

(c)

$$\begin{aligned} I_{m,9,1}(n) &= \sum_{m=1}^{n-1}m\sigma_9(m)\sigma_1(n-m) \\ &= \frac{1}{182424}n\left\{2275\sigma_{11}(n) - 691(12n-11)\sigma_9(n) - 1584\tau(n)\right\}. \end{aligned}$$

Proof. (a) By (1.2) and (1.10) we have

$$\begin{aligned}
&L(q)M^3(q) = M^3(q)\cdot L(q)\\
&= \left(1+\frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{432000}{691}\sum_{n=1}^{\infty}\tau(n)q^n\right)\left(1-24\sum_{m=1}^{\infty}\sigma_1(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{\frac{65520}{691}\sigma_{11}(N)+\frac{432000}{691}\tau(N)-24\sigma_1(N)\right.\\
&\qquad\left.-\frac{65520}{691}\cdot 24\sum_{m=1}^{N-1}\sigma_{11}(N-m)\sigma_1(m)-\frac{432000}{691}\cdot 24\sum_{m=1}^{N-1}\tau(N-m)\sigma_1(m)\right\}q^N \qquad (2.3)\\
&= 1+\sum_{N=1}^{\infty}\left\{\frac{65520}{691}\sigma_{11}(N)+\frac{432000}{691}\tau(N)-24\sigma_1(N)-\frac{65520}{691}\cdot 24\cdot I_{1,11}(N)\right.\\
&\qquad\left.-\frac{432000}{691}\cdot 24\sum_{m=1}^{N-1}\sigma_1(m)\tau(N-m)\right\}q^N.
\end{aligned}$$

So equating (2.3) with Theorem 2.2, we obtain the proof.

(b) By (1.5) and (1.15), we note that

$$\begin{aligned}
&L^2(q)M(q)N(q) = M(q)N(q)\cdot L^2(q)\\
&= \left(1-264\sum_{n=1}^{\infty}\sigma_9(n)q^n\right)\left(1-288\sum_{m=1}^{\infty}m\sigma_1(m)q^m+240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-288N\sigma_1(N)+240\sigma_3(N)-264\sigma_9(N)\right.\\
&\qquad\left.+264\cdot 288\sum_{m=1}^{N-1}\sigma_9(N-m)\cdot m\sigma_1(m)-264\cdot 240\sum_{m=1}^{N-1}\sigma_9(N-m)\sigma_3(m)\right\}q^N \qquad (2.4)\\
&= 1+\sum_{N=1}^{\infty}\{-288N\sigma_1(N)+240\sigma_3(N)-264\sigma_9(N)+264\cdot 288\cdot I_{m,1,9}(N)\\
&\qquad -264\cdot 240\cdot I_{3,9}(N)\}\,q^N.
\end{aligned}$$

Similarly, by (1.2) and Proposition 2.1 (b), we have

$$
\begin{aligned}
&L^2(q)M(q)N(q) = L(q)M(q)N(q)\cdot L(q)\\
&= \left(1 - \frac{1584}{5}\sum_{n=1}^{\infty} n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{228096}{3455}\sum_{n=1}^{\infty}\tau(n)q^n\right)\\
&\quad \times\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^m\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-\frac{1584}{5}N\sigma_9(N) + \frac{65520}{691}\sigma_{11}(N) - \frac{228096}{3455}\tau(N) - 24\sigma_1(N)\right.\\
&\quad + \frac{1584}{5}\cdot 24\sum_{m=1}^{N-1}(N-m)\,\sigma_9(N-m)\sigma_1(m) - \frac{65520}{691}\cdot 24\sum_{m=1}^{N-1}\sigma_{11}(N-m)\sigma_1(m)\\
&\quad \left. + \frac{228096}{3455}\cdot 24\sum_{m=1}^{N-1}\tau(N-m)\sigma_1(m)\right\} q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-\frac{1584}{5}N\sigma_9(N) + \frac{65520}{691}\sigma_{11}(N) - \frac{228096}{3455}\tau(N) - 24\sigma_1(N)\right.\\
&\quad + \frac{1584}{5}\cdot 24N\cdot I_{1,9}(N) - \frac{1584}{5}\cdot 24\cdot I_{m,1,9}(N) - \frac{65520}{691}\cdot 24\cdot I_{1,11}(N)\\
&\quad \left. + \frac{228096}{3455}\cdot 24\sum_{m=1}^{N-1}\sigma_1(m)\tau(N-m)\right\} q^N.
\end{aligned} \tag{2.5}
$$

Therefore, equating (2.4) with (2.5) and using Theorem 2.3 (a), we can complete the proof.

(c) We observe that

$$
\begin{aligned}
I_{m,9,1}(n) = \sum_{m=1}^{n-1} m\sigma_9(m)\sigma_1(n-m) &= \sum_{m=1}^{n-1}(n-m)\,\sigma_9(n-m)\sigma_1(m)\\
&= n\cdot I_{1,9}(n) - I_{m,1,9}(n).
\end{aligned}
$$

So we refer to Theorem 2.3 (b).

□

Theorem 2.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$
\begin{aligned}
L^2(q)M(q)N(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n + \frac{131040}{691}\sum_{n=1}^{\infty} n\sigma_{11}(n)q^n\\
- \frac{1728}{5}\sum_{n=1}^{\infty} n^2\sigma_9(n)q^n - \frac{456192}{3455}\sum_{n=1}^{\infty} n\tau(n)q^n.
\end{aligned}
$$

Proof. Insert Theorem 2.3 (b) into (2.4).

□

Next by the definition of (1.16), we can see the following proposition :

Proposition 2.2. *(See ([5], p. 155∼156)) Let $n \in \mathbb{N}$. Then we have*

(a)

$$I^2_{m,1,1}(n) = \sum_{m=1}^{n-1} m^2\sigma_1(m)\sigma_1(n-m) = \frac{1}{8}n^2\sigma_3(n) + \left(\frac{1}{24}n^2 - \frac{1}{6}n^3\right)\sigma_1(n),$$

(b)

$$I^2_{m,1,3}(n) = \sum_{m=1}^{n-1} m^2\sigma_1(m)\sigma_3(n-m) = \frac{1}{80}n^2\sigma_5(n) - \frac{1}{120}n^3\sigma_3(n) - \frac{1}{240}n^2\sigma_1(n),$$
$$I^2_{m,3,1}(n) = \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_1(n-m) = \frac{1}{24}n^2\sigma_5(n) + \left(\frac{1}{24}n^2 - \frac{1}{12}n^3\right)\sigma_3(n).$$

To be continued Proposition 2.2, we deduce Theorem 1.1 :

Proof of Theorem 1.1. (a) By (1.4) and (1.6), we have

$$\begin{aligned} &L^3(q)N(q) = N(q)\cdot L^3(q) \\ &= \left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right) \\ &\quad\times\left(1 - 1728\sum_{m=1}^{\infty} m^2\sigma_1(m)q^m + 2160\sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right) \\ &= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2016N\sigma_3(N) - 504\sigma_5(N) - 504\sigma_5(N)\right. \\ &\quad + 504\cdot 1728\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_1(m) - 504\cdot 2160\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_3(m) \\ &\quad \left. +504\cdot 504\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_5(m)\right\}q^N \\ &= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2016N\sigma_3(N) - 504\cdot 2\sigma_5(N)\right. \\ &\quad + 504\cdot 1728\sum_{m=1}^{N-1} m^2\sigma_1(m)\sigma_5(N-m) - 504\cdot 2160\cdot I_{m,3,5}(N) \\ &\quad \left. +504^2\cdot I_{5,5}(N)\right\}q^N. \end{aligned} \tag{2.6}$$

And, by (1.5) and (1.14), we obtain

$$
\begin{aligned}
&L^3(q)N(q) = L(q)N(q)\cdot L^2(q) \\
&= \left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right) \\
&\quad\times\left(1 - 288\sum_{m=1}^{\infty} m\sigma_1(m)q^m + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right) \\
&= 1 + \sum_{N=1}^{\infty}\Big\{-1008N\sigma_5(N) + 480\sigma_7(N) - 288N\sigma_1(N) \\
&\quad + 1008\cdot 288\sum_{m=1}^{N-1}(N-m)\,\sigma_5(N-m)\cdot m\sigma_1(m) \\
&\quad - 480\cdot 288\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m\sigma_1(m) + 240\sigma_3(N) \\
&\quad - 1008\cdot 240\sum_{m=1}^{N-1}(N-m)\,\sigma_5(N-m)\sigma_3(m) \\
&\quad +480\cdot 240\sum_{m=1}^{N-1}\sigma_7(N-m)\sigma_3(m)\Big\}\,q^N \\
&= 1 + \sum_{N=1}^{\infty}\{-1008N\sigma_5(N) + 480\sigma_7(N) - 288N\sigma_1(N) + 1008\cdot 288N\cdot I_{m,1,5}(N) \\
&\quad - 1008\cdot 288\sum_{m=1}^{N-1} m^2\sigma_1(m)\sigma_5(N-m) - 480\cdot 288\cdot I_{m,1,7}(N) + 240\sigma_3(N) \\
&\quad -1008\cdot 240N\cdot I_{3,5}(N) + 1008\cdot 240\cdot I_{m,3,5}(N) + 480\cdot 240\cdot I_{3,7}(N)\}\,q^N.
\end{aligned} \tag{2.7}
$$

Finally, we equate (2.6) with (2.7) and use Proposition 1.1 (c) and (d).

(b) By (1.3) and (1.12), let us consider

$$
\begin{aligned}
&L^2(q)M^2(q) = M(q)\cdot L^2(q)M(q)\\
&= \left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\\
&\quad\times\left(1+1728\sum_{m=1}^{\infty}m^2\sigma_3(m)q^m-2016\sum_{m=1}^{\infty}m\sigma_5(m)q^m+480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\Big\{1728N^2\sigma_3(N)-2016N\sigma_5(N)+480\sigma_7(N)+240\sigma_3(N)\\
&\quad+240\cdot1728\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^2\sigma_3(m)-240\cdot2016\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m\sigma_5(m)\\
&\quad+240\cdot480\sum_{m=1}^{N-1}\sigma_3(N-m)\sigma_7(m)\Big\}q^N\\
&= 1+\sum_{N=1}^{\infty}\big\{1728N^2\sigma_3(N)-2016N\sigma_5(N)+480\sigma_7(N)+240\sigma_3(N)\\
&\quad+240\cdot1728\sum_{m=1}^{N-1}m^2\sigma_3(m)\sigma_3(N-m)-240\cdot2016\cdot I_{m,5,3}(N)\\
&\quad+240\cdot480\cdot I_{3,7}(N)\big\}\,q^N.
\end{aligned}
\tag{2.8}
$$

Thus we equate (2.8) with Proposition 2.1 (a) and use Proposition 1.1 (d).

(c) Let us expand

$$
\begin{aligned}
&\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}(n-m)^2\,\sigma_5(n-m)\sigma_1(m)\\
&=n^2\cdot I_{1,5}(n)-2n\cdot I_{m,1,5}(n)+I^2_{m,1,5}(n).
\end{aligned}
$$

So we refer to Theorem 1.1 (a).

(d) First by (1.6) and (1.9), we obtain

$$
\begin{aligned}
&L^3(q)M^2(q) = M^2(q)\cdot L^3(q)\\
&= \left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\\
&\quad\times\left(1-1728\sum_{m=1}^{\infty}m^2\sigma_1(m)q^m+2160\sum_{m=1}^{\infty}m\sigma_3(m)q^m-504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\Big\{-1728N^2\sigma_1(N)+2160N\sigma_3(N)-504\sigma_5(N)+480\sigma_7(N)\\
&\quad-480\cdot1728\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m^2\sigma_1(m)+480\cdot2160\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m\sigma_3(m)\\
&\quad-480\cdot504\sum_{m=1}^{N-1}\sigma_7(N-m)\sigma_5(m)\Big\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Big\{-1728N^2\sigma_1(N)+2160N\sigma_3(N)-504\sigma_5(N)+480\sigma_7(N)\\
&\quad-480\cdot1728\sum_{m=1}^{N-1}m^2\sigma_1(m)\sigma_7(N-m)+480\cdot2160\cdot I_{m,3,7}(N)\\
&\quad-480\cdot504\cdot I_{5,7}(N)\Big\}q^N.
\end{aligned}
\tag{2.9}
$$

Second by (1.5) and (1.13), we have

$$\begin{aligned}
&L^3(q)M^2(q) = L(q)M^2(q)\cdot L^2(q)\\
&= \left(1+720\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n\right)\\
&\quad\times\left(1-288\sum_{m=1}^{\infty} m\sigma_1(m)q^m + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\{-288N\sigma_1(N)+240\sigma_3(N)+720N\sigma_7(N)\\
&\quad -720\cdot 288\sum_{m=1}^{N-1}(N-m)\,\sigma_7(N-m)\cdot m\sigma_1(m)\\
&\quad +720\cdot 240\sum_{m=1}^{N-1}(N-m)\,\sigma_7(N-m)\sigma_3(m)-264\sigma_9(N)\\
&\quad +264\cdot 288\sum_{m=1}^{N-1}\sigma_9(N-m)\cdot m\sigma_1(m)-264\cdot 240\sum_{m=1}^{N-1}\sigma_9(N-m)\sigma_3(m)\Bigg\}\, q^N\\
&= 1+\sum_{N=1}^{\infty}\{-288N\sigma_1(N)+240\sigma_3(N)+720N\sigma_7(N)-720\cdot 288N\cdot I_{m,1,7}(N)\\
&\quad +720\cdot 288\sum_{m=1}^{N-1} m^2\sigma_1(m)\sigma_7(N-m)+720\cdot 240N\cdot I_{3,7}(N)\\
&\quad -720\cdot 240\cdot I_{m,3,7}(N)-264\sigma_9(N)+264\cdot 288\cdot I_{m,1,9}(N)\\
&\quad -264\cdot 240\cdot I_{3,9}(N)\}\, q^N.
\end{aligned}\tag{2.10}$$

Therefore, we equate (2.9) and (2.10) and use Proposition 1.1 (d), Theorem 2.1 (a), and Theorem 2.3 (b).

(e) By (1.4) and (1.12), we note that

$$
\begin{aligned}
&L^2(q)M(q)N(q) = N(q)\cdot L^2(q)M(q)\\
&= \left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\\
&\quad\times\left(1 + 1728\sum_{m=1}^{\infty}m^2\sigma_3(m)q^m - 2016\sum_{m=1}^{\infty}m\sigma_5(m)q^m + 480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right)\\
&= 1 + \sum_{N=1}^{\infty}\Big\{1728N^2\sigma_3(N) - 2016N\sigma_5(N) + 480\sigma_7(N) - 504\sigma_5(N)\\
&\quad - 504\cdot 1728\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_3(m) \qquad (2.11)\\
&\quad +504\cdot 2016\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_5(m) - 504\cdot 480\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_7(m)\Big\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\big\{1728N^2\sigma_3(N) - 2016N\sigma_5(N) + 480\sigma_7(N) - 504\sigma_5(N)\\
&\quad - 504\cdot 1728\sum_{m=1}^{N-1}m^2\sigma_3(m)\sigma_5(N-m) + 504\cdot 2016\cdot I_{m,5,5}(N)\\
&\quad -504\cdot 480\cdot I_{5,7}(N)\big\}\,q^N.
\end{aligned}
$$

So we equate (2.11) with Theorem 2.4 and use Theorem 2.1 (b).

(f) We pay attention to

$$
\begin{aligned}
&\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_3(n-m)\\
&=\sum_{m=1}^{n-1}(n-m)^2\,\sigma_5(n-m)\sigma_3(m)\\
&=n^2\cdot I_{3,5}(n) - 2n\cdot I_{m,3,5}(n) + I^2_{m,3,5}(n).
\end{aligned}
$$

So we refer to Theorem 1.1 (e).

(g) We can expand

$$
\begin{aligned}
&\sum_{m=1}^{n-1}m^2\sigma_7(m)\sigma_1(n-m)\\
&=\sum_{m=1}^{n-1}(n-m)^2\,\sigma_7(n-m)\sigma_1(m)\\
&=n^2\cdot I_{1,7}(n) - 2n\cdot I_{m,1,7}(n) + I^2_{m,1,7}(n).
\end{aligned}
$$

So we refer to Theorem 1.1 (d).

□

Theorem 2.5. *For $q\in\mathbb{C}$ with $|q|<1$, we have*

(a)

$$L^3(q)N(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{4752}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + 2880\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n - 2592\sum_{n=1}^{\infty}n^3\sigma_5(n)q^n - \frac{29088}{3455}\sum_{n=1}^{\infty}\tau(n)q^n,$$

(b)

$$L^3(q)M^2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n + \frac{196560}{691}\sum_{n=1}^{\infty}n\sigma_{11}(n)q^n - \frac{5184}{5}\sum_{n=1}^{\infty}n^2\sigma_9(n)q^n + 1152\sum_{n=1}^{\infty}n^3\sigma_7(n)q^n + \frac{111744}{3455}\sum_{n=1}^{\infty}n\tau(n)q^n.$$

Proof. (a) Insert Theorem 1.1 (a) into (2.6).
(b) Insert Theorem 1.1 (d) into (2.9).

□

Now we can see that

$$I^3_{m,1,1}(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_1(n-m) = \frac{1}{12}n^3\sigma_3(n) + \left(\frac{1}{24}n^3 - \frac{1}{8}n^4\right)\sigma_1(n) \tag{2.12}$$

in ([5], p. 155), which leads us to deduce the following theorem :

Theorem 2.6. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$I^3_{m,1,3}(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_3(n-m) = \frac{1}{3360}\left\{21n^3\sigma_5(n) - 12n^4\sigma_3(n) - 14n^3\sigma_1(n) + 5\tau(n)\right\},$$

(b)

$$I^3_{m,3,1}(n) = \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_1(n-m) = \frac{1}{671}\left\{21n^3\sigma_5(n) - 4n^3\left(12n-7\right)\sigma_3(n) - \tau(n)\right\},$$

(c)

$$I^3_{m,1,5}(n) = \sum_{m=1}^{n-1} m^3\sigma_1(m)\sigma_5(n-m) = \frac{1}{3024}\left\{4n^3\sigma_7(n) - 3n^4\sigma_5(n) + 6n^3\sigma_1(n) - 7n\tau(n)\right\},$$

(d)

$$I^3_{m,3,3}(n) = \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m) = \frac{1}{720}\left\{n^3\sigma_7(n) - 3n^3\sigma_3(n) + 2n\tau(n)\right\},$$

(e)

$$\begin{aligned} I^3_{m,5,1}(n) &= \sum_{m=1}^{n-1} m^3\sigma_5(m)\sigma_1(n-m) \\ &= \frac{1}{216}\left\{4n^3\sigma_7(n) - 3n^3\left(4n-3\right)\sigma_5(n) - n\tau(n)\right\}. \end{aligned}$$

Proof. (a) First by (1.3) and (1.7), we have

$$\begin{aligned} & L^4(q)M(q) = M(q)\cdot L^4(q) \\ &= \left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1-6912\sum_{m=1}^{\infty}m^3\sigma_1(m)q^m + 10368\sum_{m=1}^{\infty}m^2\sigma_3(m)q^m\right. \\ &\qquad\qquad \left. -4032\sum_{m=1}^{\infty}m\sigma_5(m)q^m + 480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right) \\ &= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\right. \\ &\quad + 240\sigma_3(N) - 240\cdot 6912\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^3\sigma_1(m) \\ &\quad + 240\cdot 10368\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^2\sigma_3(m) - 240\cdot 4032\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m\sigma_5(m) \\ &\quad \left. +240\cdot 480\sum_{m=1}^{N-1}\sigma_3(N-m)\sigma_7(m)\right\}q^N \\ &= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\right. \\ &\quad + 240\sigma_3(N) - 240\cdot 6912\sum_{m=1}^{N-1}m^3\sigma_1(m)\sigma_3(N-m) + 240\cdot 10368\cdot I^2_{m,3,3}(n) \\ &\quad \left. -240\cdot 4032\cdot I_{m,5,3}(N) + 240\cdot 480\cdot I_{3,7}(N)\right\}q^N. \end{aligned} \tag{2.13}$$

Second by (1.6) and (1.11), we obtain

$$
\begin{aligned}
&L^4(q)M(q) = L(q)M(q)\cdot L^3(q)\\
&= \left(1 + 720\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\\
&\quad\times\left(1 - 1728\sum_{m=1}^{\infty} m^2\sigma_1(m)q^m + 2160\sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 720N\sigma_3(N)\right.\\
&\quad - 720\cdot 1728\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\cdot m^2\sigma_1(m)\\
&\quad + 720\cdot 2160\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\cdot m\sigma_3(m)\\
&\quad - 720\cdot 504\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\sigma_5(m) - 504\sigma_5(N) \qquad (2.14)\\
&\quad + 504\cdot 1728\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_1(m) - 504\cdot 2160\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_3(m)\\
&\quad \left. +504\cdot 504\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_5(m)\right\} q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) + 720N\sigma_3(N)\right.\\
&\quad - 720\cdot 1728N\cdot I^2_{m,1,3}(n) + 720\cdot 1728\sum_{m=1}^{N-1} m^3\sigma_1(m)\sigma_3(N-m)\\
&\quad + 720\cdot 2160N\cdot I_{m,3,3}(N) - 720\cdot 2160\cdot I^2_{m,3,3}(n) - 720\cdot 504N\cdot I_{3,5}(N)\\
&\quad + 720\cdot 504\cdot I_{m,5,3}(N) - 504\sigma_5(N) + 504\cdot 1728\cdot I^2_{m,1,5}(N)\\
&\quad \left. -504\cdot 2160\cdot I_{m,3,5}(N) + 504^2\cdot I_{5,5}(N)\right\} q^N.
\end{aligned}
$$

Consequently, we equate (2.13) and (2.14) and use Proposition 1.1 (c), (d), Theorem 1.1 (a), (b) and Proposition 2.2 (b).

(b) We can expand

$$
\begin{aligned}
&\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_1(n-m)\\
&= \sum_{m=1}^{n-1}(n-m)^3\,\sigma_3(n-m)\sigma_1(m)\\
&= n^3\cdot I_{1,3}(n) - 3n^2\cdot I_{m,1,3}(n) + 3n\cdot I^2_{m,1,3}(n) - I^3_{m,1,3}(n).
\end{aligned}
$$

So we refer to Proposition 1.1 (b), Proposition 2.2 (b), and Theorem 2.6 (a).

(c) By (1.4) and (1.7), we obtain

$$
\begin{aligned}
&L^4(q)N(q) = N(q)\cdot L^4(q)\\
&= \left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1-6912\sum_{m=1}^{\infty}m^3\sigma_1(m)q^m+10368\sum_{m=1}^{\infty}m^2\sigma_3(m)q^m\right.\\
&\qquad\qquad \left.-4032\sum_{m=1}^{\infty}m\sigma_5(m)q^m+480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\Big\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\\
&\quad -504\sigma_5(N)+504\cdot 6912\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^3\sigma_1(m)\\
&\quad -504\cdot 10368\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_3(m)+504\cdot 4032\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_5(m)\\
&\quad -504\cdot 480\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_7(m)\Big\}q^N\\
&= 1+\sum_{N=1}^{\infty}\big\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\\
&\quad -504\sigma_5(N)+504\cdot 6912\sum_{m=1}^{N-1}m^3\sigma_1(m)\sigma_5(N-m)-504\cdot 10368\cdot I^2_{m,3,5}(N)\\
&\quad +504\cdot 4032\cdot I_{m,5,5}(N)-504\cdot 480\cdot I_{5,7}(N)\big\}q^N.
\end{aligned}
\tag{2.15}
$$

Also by (1.6) and (1.14), we have

$$
\begin{aligned}
&L^4(q)N(q) = L(q)N(q)\cdot L^3(q)\\
&= \left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\\
&\quad\times\left(1 - 1728\sum_{m=1}^{\infty} m^2\sigma_1(m)q^m + 2160\sum_{m=1}^{\infty} m\sigma_3(m)q^m - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right)\\
&= 1 + \sum_{N=1}^{\infty}\Big\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 1008N\sigma_5(N)\\
&\quad + 1008\cdot 1728\sum_{m=1}^{N-1}(N-m)\,\sigma_5(N-m)\cdot m^2\sigma_1(m)\\
&\quad - 1008\cdot 2160\sum_{m=1}^{N-1}(N-m)\,\sigma_5(N-m)\cdot m\sigma_3(m)\\
&\quad + 1008\cdot 504\sum_{m=1}^{N-1}(N-m)\,\sigma_5(N-m)\sigma_5(m) + 480\sigma_7(N) \qquad (2.16)\\
&\quad - 480\cdot 1728\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m^2\sigma_1(m) + 480\cdot 2160\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m\sigma_3(m)\\
&\quad -480\cdot 504\sum_{m=1}^{N-1}\sigma_7(N-m)\sigma_5(m)\Big\}\,q^N\\
&= 1 + \sum_{N=1}^{\infty}\big\{-1728N^2\sigma_1(N) + 2160N\sigma_3(N) - 504\sigma_5(N) - 1008N\sigma_5(N)\\
&\quad + 1008\cdot 1728N\cdot I^2_{m,1,5}(N) - 1008\cdot 1728\sum_{m=1}^{N-1} m^3\sigma_1(m)\sigma_5(N-m)\\
&\quad - 1008\cdot 2160N\cdot I_{m,3,5}(N) + 1008\cdot 2160\cdot I^2_{m,3,5}(N) + 1008\cdot 504N\cdot I_{5,5}(N)\\
&\quad - 1008\cdot 504\cdot I_{m,5,5}(N) + 480\sigma_7(N) - 480\cdot 1728\cdot I^2_{m,1,7}(N)\\
&\quad +480\cdot 2160\cdot I_{m,3,7}(N) - 480\cdot 504\cdot I_{5,7}(N)\big\}\,q^N.
\end{aligned}
$$

So we equate (2.15) with (2.16) and refer to Proposition 1.1 (d), Theorem 1.1 (a), (d), (e), Theorem 2.1 (a) and (b).

(d) We can consider

$$
\begin{aligned}
&\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m)\\
&= \sum_{m=1}^{n-1}(n-m)^3\,\sigma_3(n-m)\sigma_3(m)\\
&= n^3\cdot I_{3,3}(n) - 3n^2\cdot I_{m,3,3}(n) + 3n\cdot I^2_{m,3,3}(n) - \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m),
\end{aligned}
$$

which shows that

$$\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m) = \frac{1}{2}\left\{n^3 \cdot I_{3,3}(n) - 3n^2 \cdot I_{m,3,3}(n) + 3n \cdot I^2_{m,3,3}(n)\right\}.$$

So we use Proposition 1.1 (c) and Theorem 1.1 (b).

(e) It is similar to proof of Theorem 2.6 (b), that is, we can expand

$$\begin{aligned}
&\sum_{m=1}^{n-1} m^3\sigma_5(m)\sigma_1(n-m)\\
&= \sum_{m=1}^{n-1} (n-m)^3\,\sigma_5(n-m)\sigma_1(m)\\
&= n^3 \cdot I_{1,5}(n) - 3n^2 \cdot I_{m,1,5}(n) + 3n \cdot I^2_{m,1,5}(n) - I^3_{m,1,5}(n).
\end{aligned}$$

So we refer to Proposition 1.1 (c), Theorem 1.1 (a), and Theorem 2.6 (c).

□

Theorem 2.7. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$\begin{aligned}
L^4(q)M(q) = 1 &+ \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{6336}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + 5760\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n\\
&- 10368\sum_{n=1}^{\infty}n^3\sigma_5(n)q^n + \frac{41472}{7}\sum_{n=1}^{\infty}n^4\sigma_3(n)q^n - \frac{4608}{24185}\sum_{n=1}^{\infty}\tau(n)q^n,
\end{aligned}$$

(b)

$$\begin{aligned}
L^4(q)N(q) = 1 &- 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n + \frac{262080}{691}\sum_{n=1}^{\infty}n\sigma_{11}(n)q^n - \frac{10368}{5}\sum_{n=1}^{\infty}n^2\sigma_9(n)q^n\\
&+ 4608\sum_{n=1}^{\infty}n^3\sigma_7(n)q^n - 3456\sum_{n=1}^{\infty}n^4\sigma_5(n)q^n - \frac{116352}{3455}\sum_{n=1}^{\infty}n\tau(n)q^n.
\end{aligned}$$

Proof. (a) Insert Theorem 2.6 (a) into (2.13).

(b) Insert Theorem 2.6 (c) into (2.15).

□

Theorem 2.8. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
I^4_{m,1,1}(n) &= \sum_{m=1}^{n-1} m^4\sigma_1(m)\sigma_1(n-m)\\
&= \frac{1}{840}\left\{50n^4\sigma_3(n) - 7n^4\,(12n-5)\,\sigma_1(n) - \tau(n)\right\},
\end{aligned}$$

(b)

$$\begin{aligned} I^4_{m,1,3}(n) &= \sum_{m=1}^{n-1} m^4 \sigma_1(m)\sigma_3(n-m) \\ &= \frac{1}{10080}\left\{35n^4\sigma_5(n) - 18n^5\sigma_3(n) - 42n^4\sigma_1(n) + 25n\tau(n)\right\}, \end{aligned}$$

(c)

$$\begin{aligned} I^4_{m,3,1}(n) &= \sum_{m=1}^{n-1} m^4 \sigma_3(m)\sigma_1(n-m) \\ &= \frac{1}{288}\left\{7n^4\sigma_5(n) - 6n^4\left(3n-2\right)\sigma_3(n) - n\tau(n)\right\}. \end{aligned}$$

Proof. (a) Now by (1.2) and (1.8), we have

$$\begin{aligned} L^6(q) &= L(q)\cdot L^5(q) \\ &= \left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1 - 20736\sum_{m=1}^{\infty} m^4\sigma_1(m)q^m + 34560\sum_{m=1}^{\infty} m^3\sigma_3(m)q^m\right. \\ &\quad \left. -17280\sum_{m=1}^{\infty} m^2\sigma_5(m)q^m + 3600\sum_{m=1}^{\infty} m\sigma_7(m)q^m - 264\sum_{m=1}^{\infty}\sigma_9(m)q^m\right) \\ &= 1 + \sum_{N=1}^{\infty}\left\{-20736N^4\sigma_1(N) + 34560N^3\sigma_3(N) - 17280N^2\sigma_5(N) + 3600N\sigma_7(N)\right. \\ &\quad - 264\sigma_9(N) - 24\sigma_1(N) + 24\cdot 20736\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^4\sigma_1(m) \\ &\quad - 24\cdot 34560\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^3\sigma_3(m) + 24\cdot 17280\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^2\sigma_5(m) \\ &\quad \left. -24\cdot 3600\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m\sigma_7(m) + 24\cdot 264\sum_{m=1}^{N-1}\sigma_1(N-m)\sigma_9(m)\right\} q^N \\ &= 1 + \sum_{N=1}^{\infty}\left\{-20736N^4\sigma_1(N) + 34560N^3\sigma_3(N) - 17280N^2\sigma_5(N) + 3600N\sigma_7(N)\right. \\ &\quad - 264\sigma_9(N) - 24\sigma_1(N) + 24\cdot 20736\sum_{m=1}^{N-1} m^4\sigma_1(m)\sigma_1(N-m) \\ &\quad - 24\cdot 34560\cdot I^3_{m,3,1}(N) + 24\cdot 17280\cdot I^2_{m,5,1}(N) - 24\cdot 3600\cdot I_{m,7,1}(N) \\ &\quad \left. +24\cdot 264\cdot I_{1,9}(N)\right\} q^N. \end{aligned} \tag{2.17}$$

And by (1.5) and (1.7), we obtain

$$
\begin{aligned}
L^6(q) &= L^2(q)\cdot L^4(q)\\
&= \left(1-288\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1-6912\sum_{m=1}^{\infty} m^3\sigma_1(m)q^m\right.\\
&\quad \left.+10368\sum_{m=1}^{\infty} m^2\sigma_3(m)q^m - 4032\sum_{m=1}^{\infty} m\sigma_5(m)q^m + 480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\right.\\
&\quad -288N\sigma_1(N)+288\cdot 6912\sum_{m=1}^{N-1}(N-m)\,\sigma_1(N-m)\cdot m^3\sigma_1(m)\\
&\quad -288\cdot 10368\sum_{m=1}^{N-1}(N-m)\,\sigma_1(N-m)\cdot m^2\sigma_3(m)\\
&\quad +288\cdot 4032\sum_{m=1}^{N-1}(N-m)\,\sigma_1(N-m)\cdot m\sigma_5(m)\\
&\quad -288\cdot 480\sum_{m=1}^{N-1}(N-m)\,\sigma_1(N-m)\sigma_7(m)+240\sigma_3(N)\\
&\quad -240\cdot 6912\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^3\sigma_1(m)+240\cdot 10368\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^2\sigma_3(m)\\
&\quad \left.-240\cdot 4032\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m\sigma_5(m)+240\cdot 480\sum_{m=1}^{N-1}\sigma_3(N-m)\sigma_7(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N)+10368N^2\sigma_3(N)-4032N\sigma_5(N)+480\sigma_7(N)\right.\\
&\quad -288N\sigma_1(N)+288\cdot 6912N\cdot I^3_{m,1,1}(N)-288\cdot 6912\sum_{m=1}^{N-1}m^4\sigma_1(m)\sigma_1(N-m)\\
&\quad -288\cdot 10368N\cdot I^2_{m,3,1}(N)+288\cdot 10368\cdot I^3_{m,3,1}(N)+288\cdot 4032N\cdot I_{m,5,1}(N)\\
&\quad -288\cdot 4032\cdot I^2_{m,5,1}(N)-288\cdot 480N\cdot I_{1,7}(N)+288\cdot 480\cdot I_{m,7,1}(N)\\
&\quad +240\sigma_3(N)-240\cdot 6912\cdot I^3_{m,1,3}(N)+240\cdot 10368\cdot I^2_{m,3,3}(N)\\
&\quad \left.-240\cdot 4032\cdot I_{m,5,3}(N)+240\cdot 480\cdot I_{3,7}(N)\right\}q^N.
\end{aligned}
\tag{2.18}
$$

So we equate (2.17) and (2.18) and use Proposition 1.1 (c), (d), Theorem 1.1 (b), (c), Proposition 2.2 (b),(2.12), Theorem 2.6 (a) and (b).

(b) First by (1.3) and (1.8), we have

$$
\begin{aligned}
&L^5(q)M(q) = M(q)\cdot L^5(q)\\
&= \left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1-20736\sum_{m=1}^{\infty}m^4\sigma_1(m)q^m+34560\sum_{m=1}^{\infty}m^3\sigma_3(m)q^m\right.\\
&\qquad\left.-17280\sum_{m=1}^{\infty}m^2\sigma_5(m)q^m+3600\sum_{m=1}^{\infty}m\sigma_7(m)q^m-264\sum_{m=1}^{\infty}\sigma_9(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-20736N^4\sigma_1(N)+34560N^3\sigma_3(N)-17280N^2\sigma_5(N)+3600N\sigma_7(N)\right.\\
&\qquad-264\sigma_9(N)+240\sigma_3(N)-240\cdot 20736\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^4\sigma_1(m)\\
&\qquad+240\cdot 34560\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^3\sigma_3(m)\\
&\qquad-240\cdot 17280\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m^2\sigma_5(m)\\
&\qquad\left.+240\cdot 3600\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m\sigma_7(m)-240\cdot 264\sum_{m=1}^{N-1}\sigma_3(N-m)\sigma_9(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-20736N^4\sigma_1(N)+34560N^3\sigma_3(N)-17280N^2\sigma_5(N)+3600N\sigma_7(N)\right.\\
&\qquad-264\sigma_9(N)+240\sigma_3(N)-240\cdot 20736\sum_{m=1}^{N-1}m^4\sigma_1(m)\sigma_3(N-m)\\
&\qquad+240\cdot 34560\cdot I^3_{m,3,3}(N)-240\cdot 17280\cdot I^2_{m,5,3}(N)+240\cdot 3600\cdot I_{m,7,3}(N)\\
&\qquad\left.-240\cdot 264\cdot I_{3,9}(N)\right\}q^N.
\end{aligned}
\tag{2.19}
$$

Second by (1.7) and (1.11) we obtain

$$
\begin{aligned}
&L^5(q)M(q) = L(q)M(q)\cdot L^4(q)\\
&= \left(1+720\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1-6912\sum_{m=1}^{\infty} m^3\sigma_1(m)q^m\right.\\
&\quad\left.+10368\sum_{m=1}^{\infty} m^2\sigma_3(m)q^m - 4032\sum_{m=1}^{\infty} m\sigma_5(m)q^m + 480\sum_{m=1}^{\infty}\sigma_7(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N) + 10368N^2\sigma_3(N) - 4032N\sigma_5(N) + 480\sigma_7(N)\right.\\
&\quad + 720N\sigma_3(N) - 720\cdot 6912\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\cdot m^3\sigma_1(m)\\
&\quad + 720\cdot 10368\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\cdot m^2\sigma_3(m)\\
&\quad - 720\cdot 4032\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\cdot m\sigma_5(m)\\
&\quad + 720\cdot 480\sum_{m=1}^{N-1}(N-m)\,\sigma_3(N-m)\sigma_7(m) - 504\sigma_5(N)\\
&\quad + 504\cdot 6912\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^3\sigma_1(m)\\
&\quad - 504\cdot 10368\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m^2\sigma_3(m)\\
&\quad \left.+504\cdot 4032\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_5(m) - 504\cdot 480\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_7(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-6912N^3\sigma_1(N) + 10368N^2\sigma_3(N) - 4032N\sigma_5(N) + 480\sigma_7(N)\right.\\
&\quad + 720N\sigma_3(N) - 720\cdot 6912N\cdot I^3_{m,1,3}(N) + 720\cdot 6912\sum_{m=1}^{N-1} m^4\sigma_1(m)\sigma_3(N-m)\\
&\quad + 720\cdot 10368N\cdot I^2_{m,3,3}(N) - 720\cdot 10368\cdot I^3_{m,3,3}(N) - 720\cdot 4032N\cdot I_{m,5,3}(N)\\
&\quad + 720\cdot 4032\cdot I^2_{m,5,3}(N) + 720\cdot 480N\cdot I_{3,7}(N) - 720\cdot 480\cdot I_{m,7,3}(N)\\
&\quad - 504\sigma_5(N) + 504\cdot 6912\cdot I^3_{m,1,5}(N) - 504\cdot 10368\cdot I^2_{m,3,5}(N)\\
&\quad \left.+504\cdot 4032\cdot I_{m,5,5}(N) - 504\cdot 480\cdot I_{5,7}(N)\right\}q^N.
\end{aligned}
\tag{2.20}
$$

Thus we equate (2.19) and (2.20) and use Proposition 1.1 (d), Theorem 1.1 (b), (e), (f), Theorem 2.1 (b), (c), Theorem 2.6 (a), (c), and (d).

(c) We can consider

$$\begin{aligned}
&\sum_{m=1}^{n-1} m^4 \sigma_3(m)\sigma_1(n-m) \\
&= \sum_{m=1}^{n-1} (n-m)^4 \sigma_3(n-m)\sigma_1(m) \\
&= n^4 \cdot I_{1,3}(n) - 4n^3 \cdot I_{m,1,3}(n) + 6n^2 \cdot I^2_{m,1,3}(n) - 4n \cdot I^3_{m,1,3}(n) + I^4_{m,1,3}(n),
\end{aligned}$$

which needs Proposition 1.1 (b), Proposition 2.2 (b), Theorem 2.6 (a), and Theorem 2.8 (b).

□

Theorem 2.9. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$\begin{aligned}
L^6(q) = 1 &+ \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{9504}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + 14400\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n \\
&- 51840\sum_{n=1}^{\infty}n^3\sigma_5(n)q^n + \frac{622080}{7}\sum_{n=1}^{\infty}n^4\sigma_3(n)q^n - \frac{248832}{5}\sum_{n=1}^{\infty}n^5\sigma_1(n)q^n \\
&- \frac{4608}{24185}\sum_{n=1}^{\infty}\tau(n)q^n,
\end{aligned}$$

(b)

$$\begin{aligned}
L^5(q)M(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n &+ \frac{327600}{691}\sum_{n=1}^{\infty}n\sigma_{11}(n)q^n - 3456\sum_{n=1}^{\infty}n^2\sigma_9(n)q^n \\
&+ 11520\sum_{n=1}^{\infty}n^3\sigma_7(n)q^n - 17280\sum_{n=1}^{\infty}n^4\sigma_5(n)q^n \\
&+ \frac{62208}{7}\sum_{n=1}^{\infty}n^5\sigma_3(n)q^n - \frac{4608}{4837}\sum_{n=1}^{\infty}n\tau(n)q^n.
\end{aligned}$$

Proof. (a) Insert Theorem 2.8 (a) into (2.17).

(b) Insert Theorem 2.8 (b) into (2.19).

□

Theorem 2.10. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
I^5_{m,1,1}(n) &= \sum_{m=1}^{n-1} m^5 \sigma_1(m)\sigma_1(n-m) \\
&= \frac{1}{336}\left\{15n^5\sigma_3(n) - 14n^5\left(2n-1\right)\sigma_1(n) - n\tau(n)\right\}.
\end{aligned}$$

Proof. We can show that

$$\sum_{m=1}^{n-1} m^5\sigma_1(m)\sigma_1(n-m) = \sum_{m=1}^{n-1} (n-m)^5\,\sigma_1(n-m)\sigma_1(m)$$
$$= n^5 \cdot I_{1,1}(n) - 5n^4 \cdot I_{m,1,1}(n) + 10n^3 \cdot I^2_{m,1,1}(n) - 10n^2 \cdot I^3_{m,1,1}(n) + 5n \cdot I^4_{m,1,1}(n)$$
$$- \sum_{m=1}^{n-1} m^5\sigma_1(m)\sigma_1(n-m),$$

which leads that

$$\sum_{m=1}^{n-1} m^5\sigma_1(m)\sigma_1(n-m) = \frac{1}{2}\left\{n^5 \cdot I_{1,1}(n) - 5n^4 \cdot I_{m,1,1}(n) + 10n^3 \cdot I^2_{m,1,1}(n) \right.$$
$$\left. -10n^2 \cdot I^3_{m,1,1}(n) + 5n \cdot I^4_{m,1,1}(n)\right\}.$$

So we refer to Proposition 1.1 (a), Proposition 2.2 (a), (2.12), and Theorem 2.8 (a).

□

Proof of Lemma 1.2. We multiply Theorem 2.9 (a) by (1.2) :

$$
\begin{aligned}
&L^7(q) = L(q)\cdot L^6(q)\\
&= \left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1+\frac{65520}{691}\sum_{m=1}^{\infty}\sigma_{11}(m)q^m-\frac{9504}{5}\sum_{m=1}^{\infty}m\sigma_9(m)q^m\right.\\
&\quad +14400\sum_{m=1}^{\infty}m^2\sigma_7(m)q^m-51840\sum_{m=1}^{\infty}m^3\sigma_5(m)q^m+\frac{622080}{7}\sum_{m=1}^{\infty}m^4\sigma_3(m)q^m\\
&\quad \left.-\frac{248832}{5}\sum_{m=1}^{\infty}m^5\sigma_1(m)q^m-\frac{4608}{24185}\sum_{m=1}^{\infty}\tau(m)q^m\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{\frac{65520}{691}\sigma_{11}(N)-\frac{9504}{5}N\sigma_9(N)+14400N^2\sigma_7(N)-51840N^3\sigma_5(N)\right.\\
&\quad +\frac{622080}{7}N^4\sigma_3(N)-\frac{248832}{5}N^5\sigma_1(N)-\frac{4608}{24185}\tau(N)-24\sigma_1(N)\\
&\quad -24\cdot\frac{65520}{691}\sum_{m=1}^{N-1}\sigma_1(N-m)\sigma_{11}(m)+24\cdot\frac{9504}{5}\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m\sigma_9(m)\\
&\quad -24\cdot 14400\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^2\sigma_7(m)+24\cdot 51840\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^3\sigma_5(m)\\
&\quad -24\cdot\frac{622080}{7}\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^4\sigma_3(m)\\
&\quad \left.+24\cdot\frac{248832}{5}\sum_{m=1}^{N-1}\sigma_1(N-m)\cdot m^5\sigma_1(m)+24\cdot\frac{4608}{24185}\sum_{m=1}^{N-1}\sigma_1(N-m)\tau(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{\frac{65520}{691}\sigma_{11}(N)-\frac{9504}{5}N\sigma_9(N)+14400N^2\sigma_7(N)-51840N^3\sigma_5(N)\right.\\
&\quad +\frac{622080}{7}N^4\sigma_3(N)-\frac{248832}{5}N^5\sigma_1(N)-\frac{4608}{24185}\tau(N)-24\sigma_1(N)\\
&\quad -24\cdot\frac{65520}{691}\cdot I_{1,11}(N)+24\cdot\frac{9504}{5}\cdot I_{m,9,1}(N)-24\cdot 14400\cdot I^2_{m,7,1}(N)\\
&\quad +24\cdot 51840\cdot I^3_{m,5,1}(N)-24\cdot\frac{622080}{7}\cdot I^4_{m,3,1}(N)+24\cdot\frac{248832}{5}\cdot I^5_{m,1,1}(N)\\
&\quad \left.+24\cdot\frac{4608}{24185}\sum_{m=1}^{N-1}\sigma_1(N-m)\tau(m)\right\}q^N.
\end{aligned}
$$

Therefore we use Theorem 1.1 (g), Theorem 2.3 (a), (c), Theorem 2.6 (e), Theorem 2.8 (c), and Theorem 2.10.

□

3 Proof of Theorem 1.3

In this section we construct the new convolution sums by multiplying $\sigma_1(n)$ for an positive integer n into the given convolution sums in Proposition 1.1 and so we can obtain Theorem 1.3.

Proof of Theorem 1.3. Since proofs are similar, so we only prove Theorem 1.3 (j). Now by Proposition 1.1 (d), we can expand

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} a\sigma_7(a)\sigma_1(b)\sigma_1(c) = \sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k\sigma_1(k)\sigma_1(m-k)\sigma_1(n-m)\\
&= \sum_{m=1}^{n-1}\left[\frac{1}{1800}\left\{33m\sigma_9(m) - 25m\,(4m-3)\,\sigma_7(m) - 8\tau(m)\right\}\right]\sigma_1(n-m)\\
&= \frac{1}{1800}\left\{33\sum_{m=1}^{n-1} m\sigma_9(m)\sigma_1(n-m) - 25\cdot 4\sum_{m=1}^{n-1} m^2\sigma_7(m)\sigma_1(n-m)\right.\\
&\qquad \left. +25\cdot 3\sum_{m=1}^{n-1} m\sigma_7(m)\sigma_1(n-m) - 8\sum_{m=1}^{n-1}\tau(m)\sigma_1(n-m)\right\}\\
&= \frac{1}{1800}\left\{33\cdot I_{m,9,1}(n) - 25\cdot 4\cdot I^2_{m,7,1}(n) + 25\cdot 3\cdot I_{m,7,1}(n)\right.\\
&\qquad \left. -8\sum_{m=1}^{n-1}\tau(m)\sigma_1(n-m)\right\}.
\end{aligned}
$$

So we use Proposition 1.1 (d), Theorem 1.1 (g), Theorem 2.3 (a) and (c).

□

4 Conclusions

In this paper, we study some various convolution sum formulae mainly focusing on the form

$$I^k_{m,e,f}(n) = \sum_{m=1}^{n-1} m^k\sigma_e(m)\sigma_f(n-m)$$

for $k \in \mathbb{N}$ $(1 \le k \le 5)$ and an odd positive integer e and f. Moreover, we obtain some identities deduced easily from the convolution sums $I^k_{m,e,f}(n)$.

Competing Interests

The author declares that no competing interests exist.

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Appendix

The first twenty values of $\tau(n)$ are given in the Table 1.

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	-6048	11	534612	16	987136
2	-24	7	-16744	12	-370944	17	-6905934
3	252	8	84480	13	-577738	18	2727432
4	-1472	9	-113643	14	401856	19	10661420
5	4830	10	-115920	15	1217160	20	-7109760

TABLE 1. $\tau(n)$ for n $(1 \leq n \leq 20)$

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