British Journal of Mathematics & Computer Science
8(3): 246-264, 2015, Article no.BJMCS.2015.159
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

A Mathematical Model for Population Density Dynamics of Weed-Crop Competition

M. O. Nasir[1*], N. I. Akinwande[2], M. G. M. Kolo[3], J. Mohammed[2] and Rose Abbah[4]

[1]*Department of Mathematics and Statistics, Federal Polytechnic, Nasarawa, Nigeria.*
[2]*Department of Mathematics and Statistics, Federal University of Technology, Minna, Nigeria.*
[3]*Department of Crop Productions, Federal University of Technology, Minna, Nigeria.*
[4]*Department of Mathematics, University of Abuja, Abuja, Nigeria.*

Article Information

DOI: 10.9734/BJMCS/2015/15975
Editor(s):
(1) Longhua Zhao, Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, USA.
(2) Jaime Rangel-Mondragon, Faculty of Informatics, Queretaro´s Institute of Technology, Mexico and Faculty of Computer Science, Autonomous University of Querétaro, Mexico.
Reviewers:
(1) E. H. Etuk, Rivers State University of Science and Technology, Nigeria.
(2) Matej Mencinger, University of Maribor, Faculty of Civil Engineering, Slovenia.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=1032&id=6&aid=8697

Original Research Article

Received: 30 December 2014
Accepted: 27 February 2015
Published: 06 April 2015

Abstract

A model for the dynamics of homogeneous population competition between two species of weeds and a crop is formulated to gain in-sight into the behaviour of crop growing with weeds. We used assumptions based upon reasonable biological process to derive from single weed model equation, the systems of difference equations that described the dynamics of weed-crop competition. Steady- state solutions of the model are obtained and analyzed for local stability or otherwise. The results show that the extinction steady state is not stable without control and the conditions for stability of two plants steady-states are given. The weed-crop coexistence steady state of our model is locally asymptotically stable. Besides, the graphical profile of the model shows that the crop's growth may be stagnated by the weeds' densities. Hence, we conclude that the crop's growth may be stagnated, but survive in the mixture of the two species of weeds. However, application of effective control measure to eradicate the two weeds species will enhance crop growth at its optimum capacity.

Keywords: Crop–weed; competition; steady–state; coexistence; asymptotically stable.

**Corresponding author: nas_omak@yahoo.com;*

1 Introduction

Weed and Crop are both plants. Crop is a desirable plant mostly cultivated, while weed is unwanted plants that always grow naturally. Weeds are generally defined as uncultivated plant species that proliferate in agricultural setting thereby, interfering with crop production. In fact, weed is a term applied to any plant that grows in a place it is not wanted. They exist only in natural environments that have been disturbed by humans such as agricultural lands, recreational parks, and irrigation dams [1]. However, plants are very important in the natural world, since they are the survival basis for all kinds of creatures, including human being, animals and microorganism [2, 3]. The weeds of natural areas often differ from those affecting agricultural fields. The agricultural weeds are frequently annuals which colonize bare ground. Weeds of natural areas are often perennials or biennials that are capable of invading established natural areas and are hard to get rid of.

Population dynamics involve the study of population growth (numerical change in time), composition and spatial dispersion. The objectives are to identify the causes of change in population and to explain how this cause act and interact to produce the observed pattern. Not until recent past population models were concentrated mainly on the use of differential equations. Although, most populations such as weeds and organisms whose classification are based on their developmental relationships live in seasonal environments and for this reason, have yearly rhythms of reproduction (birth) and death. Besides, measurements are often made once a year because interest is centred on change in population from one season to another. Continuous-time models (differential equations) are not well appropriate to describe these types of dynamical processes. Hence, there is need for other modelling techniques, especially when interested in population with only annual reproductive tendencies or expected changes that happen seasonally. Discrete-time models are better suited for organism with annual or seasonal reproductive patterns [4,5,6,7]. Since plant has discrete generations (seasonal reproduction), difference dynamical equations are proper mathematical tool to describe the behaviour of population with no overlapping generations such as weeds. Furthermore, many researchers have paid attention in recent times to discrete-time population models, since it is governed by discrete systems which are more appropriate when the populations under consideration have non over lapping generations [8].

In this study we intend to employ biological process to develop discrete-time models for the dynamics of weed-crop competition. This research work considered homogeneous population competition between weeds and crop to understand the deleterious effects and gain in-sight into the behaviour of weeds growing with a crop plant.

2 Materials and Methods

2.1 Formulation of the Model Equations

Baseline for studies of population dynamics of weeds as with most plant species are usually on analysis of single species in defined habitats and often experimentally manipulated.

In this section, we formulate from the simple model equation proposed to described the dynamics of single weed species proliferation as stated in [9], thus

$$n_{t+1} = \frac{\beta n_t}{1 + a n_t} + \gamma n_t \ . \tag{2.1}$$

This is a non-linear difference equation for the density of mature single-species of weed.

In modelling weed-crop interaction in continuous arable cropping, we assumed that an annual crop species is sown at approximately the same density each cropping season and two species of weeds n_1 and n_2 which may be superior competitors are growing in mixture with the crop (a desirable plant). We observed that, it is rare to find a single weed species growing in a particular arable area, although one may be in abundance or dominates.

In order to study the population dynamics of weeds in mixture with crop, the single-species weed model (2.1) is first extended to a model of two weed species for the competitive effect of introducing the second species n_2 into monoculture of the first n_1. The term $a_{12}n_{2,t}$ is introduced to reflect the reduction per capita rate of growth or yield per unit area of the first species. Therefore, the dynamics of the mixture is described by the model equation;

$$n_{1,t+1} = \frac{\beta_1 n_{1,t}}{1 + a_{11}n_{1,t} + a_{12}n_{2,t}} + \gamma_1 n_{1,t}\,. \tag{2.2}$$

In the same manner, the addition of the term $a_{21}n_{1,t}$, crowding density of individuals of the first specie n_1 into monoculture of the second n_2 will lead to reduction in its growth (second species) but may not necessarily be of the same magnitude. Hence, similar model equation is

$$n_{2,t+1} = \frac{\beta_2 n_{2,t}}{1 + a_{21}n_{1,t} + a_{22}n_{2,t}} + \gamma_2 n_{2,t}\,. \tag{2.3}$$

where a_{11} and a_{22} are intra-specific coefficients (competitions) of species 1 and 2, respectively.

In order to study (account for) the effect of one species on the other in the mixture, we find the relative crowding effect of each species by normalizing (2.2) and (2.3) via changing of variables, as follows;

Substituting for $n_{1,t} = \frac{1}{a_{11}}N_{1,t}$ and $n_{2,t} = \frac{1}{a_{22}}N_{2,t}$ in (2.2) and (2.3) give

$$N_{1,t+1} = \frac{\beta_1 N_{1,t}}{1 + N_{1,t} + \alpha_{12}N_{2,t}} + \gamma_1 N_{1,t}\,, \tag{2.4}$$

$$N_{2,t+1} = \frac{\beta_2 N_{2,t}}{1 + \alpha_{21}N_{1,t} + N_{2,t}} + \gamma_2 N_{2,t}\,, \tag{2.5}$$

where $\alpha_{12} = \frac{a_{12}}{a_{22}}$ and $\alpha_{21} = \frac{a_{21}}{a_{11}}$.

Here α_{12} and α_{21} are inter-specific competitions effects of species 2 on 1 and 1 on 2 respectively. Therefore, equations (2.4) and (2.5) are coupled non-linear difference equations, which give inter-species model for competition between homogeneous densities of two weed-species in the absence of crop. For the crop interaction, since crop is a plant like weed but a desirable one, when the three plants are competing, their fitness will be adversely affected. Hence, a two-weed species competition model equation (2.4) and (2.5) are extended to three competing plants. The dynamics of weed proliferation in the presence of crop density is described by a system of difference equations (2.6) – (2.8);

Fig. 2.1 is a schematic representation of the model. Subsequent

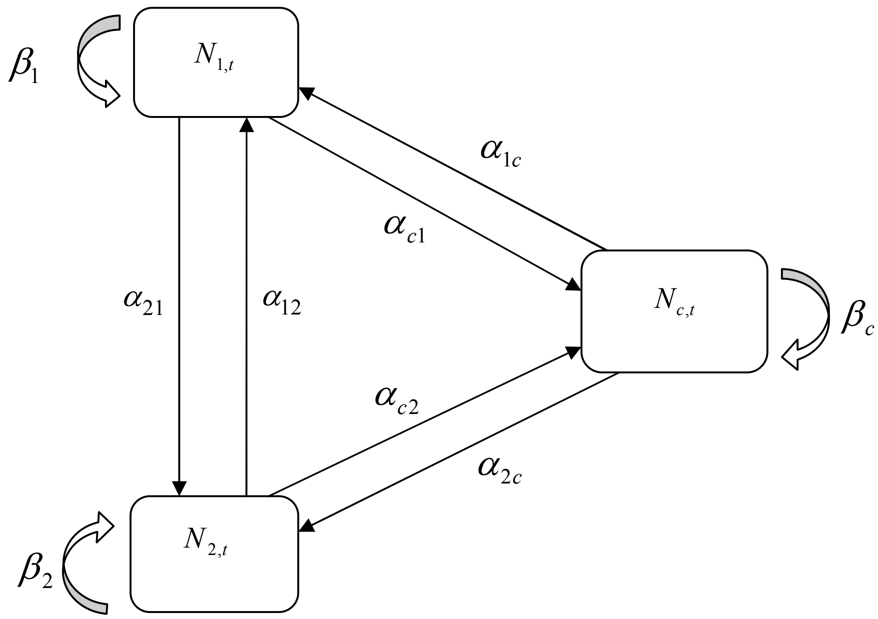

Fig. 2.1. Schematic model for weed-crop competition of homogeneous population density

$$N_{c,t+1} = \frac{\beta_c N_{c,t}}{1 + N_{c,t} + \alpha_{c1} N_{1,t} + \alpha_{c2} N_{2,t}}, \tag{2.6}$$

$$N_{1,t+1} = \frac{\beta_1 N_{1,t}}{1 + \alpha_{1c} N_{c,t} + N_{1,t} + \alpha_{12} N_{2,t}} + \gamma_1 N_{1,t}, \tag{2.7}$$

$$N_{2,t+1} = \frac{\beta_2 N_{2,t}}{1 + \alpha_{2c} N_{c,t} + \alpha_{21} N_{1,t} + N_{2,t}} + \gamma_2 N_{2,t}, \tag{2.8}$$

where $N_{c,t}$ is the density of established crop seed sown in the growing season, *t* , which survived to maturity and produce seeds for the next season, $N_{1,t}$ density of mature weed species 1 in the growing season *t* and $N_{2,t}$ is the density of mature weed species 2 in the growing season *t*. Here, β_c, β_1 and β_2 are the recruitment factors (the growth factors) of crop and two weed species respectively. α_{ic} and α_{ij} are competition coefficients of crop and weeds respectively, while γ_i are the residual seeds of the weeds.

The following assumptions are taken into consideration in the formulation of the model equations for the above ground competitions among established homogeneous densities of two species of weed and a crop.

1. There are enough growth resources; e.g. nutrients, light, and water that promote continuous growth of at least two plant species.
2. Between the two populations of weed and crop there are inter- specific competitions.
3. All the dormant seeds are viable and no predation by insects,
4. All the crop seeds germinated (so there are no dormant or residual crop seeds)
5. All parameters involved with the model formulation are non-negatives.

The resulting model equations (2.6) – (2.8) were to establish the conditions in which a crop could coexist with two species of weed through their competitive interactions for space with each other (since it was assumed that other growth resources are enough).

2.2 Dynamical Behaviour of the Developed Weed-Crop Competition Model

2.2.1 Steady-state solutions and stability analysis

The goal of this section is to employ analytic method to describe the nature and behaviour of our model for the two-species of weeds competing with a crop. This will be done without actually solving or approximating them, since in contrast with differential equations, the existence and uniqueness of solutions of difference equations initial value problems are always guaranteed [10]. Usually in the investigation of any dynamical system a primary step is to find its fixed or steady state solutions. So, for the steady states (fixed points) of our three-species competition model equations (2.6) – (2.8), there are seven steady states of interest; three single-species steady states where one species is at carrying capacity and other two are locally extinct, three two-species steady states when two species co-exist and the third is absent (extinct) and finally, one three-species steady state when co-existence among all the three species is possible. To obtain these steady-state solutions, we let $E(\overline{N}_c, \overline{N}_1, \overline{N}_2)$ be the solutions of the system (2.6) – (2.8). So, zero steady state $E_0(0, 0, 0)$ always exist. For non-zero positive steady state, we let $N_{i,t+1} = N_{i,t} = \overline{N_i}$ in (2.6) - (2.8). So, we have

$$\overline{N}_c + \alpha_{c1}\overline{N}_1 + \alpha_{c2}\overline{N}_2 = \beta_c - 1, \tag{2.9}$$

$$\alpha_{1c}\overline{N}_c + \overline{N}_1 + \alpha_{12}\overline{N}_2 = \theta, \tag{2.10}$$

$$\alpha_{2c}\overline{N}_c + \alpha_{21}\overline{N}_1 + \overline{N}_2 = \phi, \tag{2.11}$$

where $\theta = \dfrac{\beta_1}{1-\gamma_1} - 1$ and $\phi = \dfrac{\beta_2}{1-\gamma_2} - 1$. Here, θ and ϕ are the maximum steady-state populations of each weed species in the mixture without a control.

The steady-state solutions of interest when two species co-exist while the third is locally extinct and when co-existence among all the three species is possible were obtained next.

2.2.2 Two- weed species dominance (infestation)

If $\overline{N}_1 \neq 0,$ and $\overline{N}_2 \neq 0$ but $\overline{N}_c = 0$, this is equivalent to proliferation of two-species of weed without a crop. Now equations (2.10) and (2.11) become

$$\overline{N}_1 + \alpha_{12}\overline{N}_2 = \theta, \tag{2.12}$$

$$\alpha_{21}\overline{N}_1 + \overline{N}_2 = \phi. \tag{2.13}$$

Solving equations (2.12) and (2.13) simultaneously, gives the non-negative steady state as

$$E_1\left(0, \frac{\theta - \alpha_{12}\phi}{(1-\alpha_{12}\alpha_{21})}, \frac{\phi - \theta\alpha_{21}}{(1-\alpha_{12}\alpha_{21})}\right). \tag{2.14}$$

E_1 exists provided $\theta = \phi$ and $\alpha_{12} \neq \alpha_{21}$ such that $\alpha_{12}\alpha_{21} < 1,$ $\alpha_{12}\phi < \theta$ and $\theta < \dfrac{\phi}{\alpha_{21}}$.

Here, $\theta = \phi$ implies that the two weed species have equal possibility to attain their maximum densities at the steady state. So, the population density dynamics of the two weeds species depend on inter-species competition coefficients, such that $\alpha_{12} < 1$ and $\alpha_{21} < 1$ where $\alpha_{12} \neq \alpha_{21}$.

2.2.3 Crop and weed species dominance

If $\overline{N}_2 = 0$ and $\overline{N}_1 \neq 0, \overline{N}_c \neq 0,$ equations (2.9) and (2.10) become

$$\overline{N}_c + \alpha_{c1}\overline{N}_1 = \beta_c - 1. \tag{2.15}$$

$$\alpha_{1c}\overline{N}_c + \overline{N}_1 = \theta. \tag{2.16}$$

Solving equations (2.15) and (2.16) simultaneously, to obtain the non-zero steady state

$$E_2\left(\frac{\beta_c - (1+\alpha_{c1}\theta)}{(1-\alpha_{c1}\alpha_{1c})}, \frac{\theta - \alpha_{1c}(\beta_c - 1)}{(1-\alpha_{c1}\alpha_{1c})}, 0\right). \tag{2.17}$$

E_2 exists provided $\alpha_{c1}\alpha_{1c} < 1,$ $1 + \alpha_{c1}\theta < \beta_c$ and $\beta_c < \dfrac{(\theta + \alpha_{1c})}{\alpha_{1c}}$.

Also, from equations (2.9) and (2.11) when $\overline{N}_c \neq 0,$ $\overline{N}_2 \neq 0$, but $\overline{N}_1 = 0$, we have the non-zero steady state

$$E_3\left(\frac{\beta_c - (1+\alpha_{c2}\phi)}{(1-\alpha_{c2}\alpha_{2c})}, 0, \frac{\phi - \alpha_{2c}(\beta_c - 1)}{(1-\alpha_{c2}\alpha_{2c})}\right). \tag{2.18}$$

E_3 exists provided $\alpha_{c2}\alpha_{2c} < 1$, $1+\alpha_{c2}\phi < \beta_c$ and $\beta_c < \dfrac{\phi+\alpha_{2c}}{\alpha_{2c}}$.

2.2.4 Crop coexistence with two-weed species

The fixed points are obtained when a crop survives (grows) in the presence of two species of weeds. At non-zero steady states, that is when $\overline{N}_c \neq 0$, $\overline{N}_1 \neq 0$ and $\overline{N}_2 \neq 0$ equations (2.9) - (2.11) written in matrix form

$$\begin{pmatrix} 1 & \alpha_{c1} & \alpha_{c2} \\ \alpha_{1c} & 1 & \alpha_{12} \\ \alpha_{2c} & \alpha_{21} & 1 \end{pmatrix}\begin{pmatrix} \overline{N}_c \\ \overline{N}_1 \\ \overline{N}_2 \end{pmatrix} = \begin{pmatrix} \beta_c - 1 \\ \theta \\ \phi \end{pmatrix}. \tag{2.19}$$

This is equivalent to matrix equation $A\overline{N} = B$. The values of $\overline{N}_c$, $\overline{N}_1$ and $\overline{N}_2$ are obtained from (2.19) as

$$\left.\begin{aligned} \overline{N}_c &= \frac{\beta_c(1-\alpha_{12}\alpha_{21}) - \theta(\alpha_{c1}-\alpha_{c2}\alpha_{21}) - \phi(\alpha_{c2}-\alpha_{c1}\alpha_{12})}{(1-\alpha_{12}\alpha_{21}) - \alpha_{1c}(\alpha_{c1}-\alpha_{c2}\alpha_{21}) - \alpha_{2c}(\alpha_{c2}-\alpha_{c1}\alpha_{12})} \\ \overline{N}_1 &= \frac{\theta(1-\alpha_{c2}\alpha_{2c}) - \beta_c(\alpha_{1c}-\alpha_{12}\alpha_{2c}) - \phi(\alpha_{12}-\alpha_{1c}\alpha_{c2})}{(1-\alpha_{12}\alpha_{21}) - \alpha_{1c}(\alpha_{c1}-\alpha_{c2}\alpha_{21}) - \alpha_{2c}(\alpha_{c2}-\alpha_{c1}\alpha_{12})} \\ \overline{N}_2 &= \frac{\phi(1-\alpha_{c1}\alpha_{1c}) - \beta_c(\alpha_{2c}-\alpha_{1c}\alpha_{21}) - \theta(\alpha_{21}-\alpha_{c1}\alpha_{2c})}{(1-\alpha_{12}\alpha_{21}) - \alpha_{1c}(\alpha_{c1}-\alpha_{c2}\alpha_{21}) - \alpha_{2c}(\alpha_{c2}-\alpha_{c1}\alpha_{12})} \end{aligned}\right\}. \tag{2.20}$$

So, the non-zero steady state $E_4(\overline{N}_c, \overline{N}_1, \overline{N}_2)$ as given in (2.20) exists provided the following conditions hold $\alpha_{12}\alpha_{21} < 1$, $\alpha_{c1} < \alpha_{c2}\alpha_{21}$, $\alpha_{c2} < \alpha_{c1}\alpha_{12}$, $\alpha_{c2}\alpha_{2c} < 1$, $\alpha_{1c} < \alpha_{12}\alpha_{2c}$, $\alpha_{12} < \alpha_{1c}\alpha_{c2}$, $\alpha_{c1}\alpha_{1c} < 1$, $\alpha_{2c} < \alpha_{1c}\alpha_{21}$ and $\alpha_{21} < \alpha_{c1}\alpha_{2c}$ besides $\beta_c, \theta, \phi > 0$.

The conditions are summarized in Table 2.1.

2.2.5 Local stability of the steady states for the model equations of weed-crop competition

In this section, Perron's approach to the stability analysis is adapted to our model equation. Hence, The linearization of system (2.6) – (2.8) about the steady state $E(\overline{N}_c, \overline{N}_1, \overline{N}_2)$ yields the partial derivative matrix (variation matrix).

$$D(\overline{N}_c,\overline{N}_1,\overline{N}_2)=\begin{pmatrix} \dfrac{\beta_c(1+\alpha_{c1}\overline{N}_1+\alpha_{c2}\overline{N}_2)}{(1+\overline{N}_c+\alpha_{c1}\overline{N}_1+\alpha_{c2}\overline{N}_2)^2} & \dfrac{-\beta_c\alpha_{c1}\overline{N}_c}{(1+\overline{N}_c+\alpha_{c1}\overline{N}_1+\alpha_{c2}\overline{N}_2)^2} & \dfrac{-\beta_c\alpha_{c2}\overline{N}_c}{(1+\overline{N}_c+\alpha_{c1}\overline{N}_1+\alpha_{c2}\overline{N}_2)^2} \\ \dfrac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\alpha_{1c}\overline{N}_c+\overline{N}_1+\alpha_{12}\overline{N}_2)^2} & \dfrac{\beta_1(1+\alpha_{1c}\overline{N}_c+\alpha_{12}\overline{N}_2)}{(1+\alpha_{1c}\overline{N}_c+\overline{N}_1+\alpha_{12}\overline{N}_2)^2}+\gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\alpha_{1c}\overline{N}_c+\overline{N}_1+\alpha_{12}\overline{N}_2)^2} \\ \dfrac{-\alpha_{2c}\beta_2\overline{N}_2}{(1+\alpha_{2c}\overline{N}_c+\alpha_{21}\overline{N}_1+\overline{N}_2)^2} & \dfrac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\alpha_{2c}\overline{N}_c+\alpha_{21}\overline{N}_1+\overline{N}_2)^2} & \dfrac{\beta_2(1+\alpha_{2c}\overline{N}_c+\alpha_{21}\overline{N}_1)}{(1+\alpha_{2c}\overline{N}_c+\alpha_{21}\overline{N}_1+\overline{N}_2)^2}+\gamma_2 \end{pmatrix} \tag{2.21}$$

Table 2.1. Conditions for existence of steady state solutions

	Solution co-ordinates	**Conditions for existence**
E_0	(0, 0, 0)	$\beta_c = \beta_1 = \beta_2 = 0$
E_1	$\left(0,\dfrac{\theta-\alpha_{12}\phi}{(1-\alpha_{12}\alpha_{21})},\dfrac{\phi-\theta\alpha_{21}}{(1-\alpha_{12}\alpha_{21})}\right)$	$\alpha_{12}\alpha_{21}<1, \alpha_{12}\phi<\theta$, $\theta<\dfrac{\phi}{\alpha_{21}}$, . $\alpha_{12}\neq\alpha_{21}$ and $\theta=\phi$.
E_2	$\left(\dfrac{\beta_c-(1+\alpha_{c1}\theta)}{(1-\alpha_{c1}\alpha_{1c})},\dfrac{\theta-\alpha_{1c}(\beta_c-1)}{(1-\alpha_{c1}\alpha_{1c})},0\right)$	$\alpha_{c1}\alpha_{1c}<1,\ 1+\alpha_{c1}\theta<\beta_c$ and $\beta_c<\dfrac{(\theta+\alpha_{1c})}{\alpha_{1c}}$.
E_3	$\left(\dfrac{\beta_c-(1+\alpha_{c2}\phi)}{(1-\alpha_{c2}\alpha_{2c})},0,\dfrac{\phi-\alpha_{2c}(\beta_c-1)}{(1-\alpha_{c2}\alpha_{2c})}\right)$	$\alpha_{c2}\alpha_{2c}<1,\ 1+\alpha_{c2}\phi<\beta_c$ and $\beta_c<\dfrac{\phi+\alpha_{2c}}{\alpha_{2c}}$.
E_4	$(\overline{N}_c,\overline{N}_1,\overline{N}_2)$	$\alpha_{12}\alpha_{21}<1,\ \alpha_{c1}<\alpha_{c2}\alpha_{21}$, $\alpha_{c2}<\alpha_{c1}\alpha_{12}$ $\alpha_{c2}\alpha_{2c}<1,\ \alpha_{1c}<\alpha_{12}\alpha_{2c}$, $\alpha_{12}<\alpha_{1c}\alpha_{c2}$, $\alpha_{c1}\alpha_{1c}<1$, $\alpha_{2c}<\alpha_{1c}\alpha_{21}$ and $\alpha_{21}<\alpha_{c1}\alpha_{2c}$.

2.2.6 Stability of the extinction steady state *E*$_0$ (0,0,0)

Evaluating (2.21) at E_0 (0,0,0) gives

$$D(E_0)=\begin{pmatrix} \beta_c & 0 & 0 \\ 0 & \beta_1+\gamma_1 & 0 \\ 0 & 0 & \beta_2+\gamma_2 \end{pmatrix}. \tag{2.22}$$

The eigenvalues of $D(E_0)$ are the main diagonal elements.

So, the stability of E_0 depends on $\beta_c < 1$, $\beta_1 + \gamma_1 < 1$ and $\beta_2 + \gamma_2 < 1$. However, from the conditions for the existence of the steady states $\beta_c > 1$, $\beta_1 + \gamma_1 > 1$ and $\beta_2 + \gamma_2 > 1$. Hence, E_0 is unstable without control.

2.2.7 Stability of the steady state for two species of weed dominance

For the coexistence steady state of the two-species of weeds, (2.21) evaluated at $E_1(0, \overline{N}_1, \overline{N}_2)$ gives

$$D(E_1) = \begin{pmatrix} \dfrac{\beta_c}{(1+\alpha_{c1}\overline{N}_1 + \alpha_{c2}\overline{N}_2)^2} & 0 & 0 \\ \dfrac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\overline{N}_1 + \alpha_{12}\overline{N}_2)^2} & \dfrac{\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\overline{N}_1 + \alpha_{12}\overline{N}_2)^2} + \gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\overline{N}_1 + \alpha_{12}\overline{N}_2)^2} \\ \dfrac{-\alpha_{2c}\beta_2\overline{N}_2}{(1+\alpha_{21}\overline{N}_1 + \overline{N}_2)^2} & \dfrac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\alpha_{21}\overline{N}_1 + \overline{N}_2)^2} & \dfrac{\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\alpha_{21}\overline{N}_1 + \overline{N}_2)^2} + \gamma_2 \end{pmatrix}. \qquad (2.23)$$

Further simplification using (2.9) – (2.11) when $\overline{N}_c = 0$ gives

$$D(E_1) = \begin{pmatrix} \dfrac{1}{\beta_c} & 0 & 0 \\ \dfrac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\theta)^2} & \dfrac{\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\theta)^2} \\ \dfrac{-\alpha_{2c}\beta_2\overline{N}_2}{(1+\phi)^2} & \dfrac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\phi)^2} & \dfrac{\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_2 \end{pmatrix}. \qquad (2.24)$$

The matrix $D(E_1)$ in (2.24) is a block lower rectangular matrix. So, its eigenvalues are equal to the eigenvalues of the 2 x 2 and 1 x 1 matrices involved. It is observed that the eigenvalues of the 1x1 matrix which is the single element of that matrix has magnitude less than one. That is

$$\frac{1}{\beta_c} < 1 \text{ since } \beta_c > 1.$$

Therefore, the local stability analysis of the steady state E_1 depends on the analysis of the block 2x2 matrix in (2.24), thus;

$$A = \begin{pmatrix} \dfrac{\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\theta)^2} \\ \dfrac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\phi)^2} & \dfrac{\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_2 \end{pmatrix}. \qquad (2.25)$$

The stability of A is determined by the Trace - determinant analysis [11]. So,

$$tr(A) = \frac{\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \gamma_1 + \frac{\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_2 .$$

$$\det(A) = \frac{\gamma_2\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \frac{\gamma_1\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_1\gamma_2 .$$

For stability we need to show that $|trA| - 1 < \det A < 1$. To do this it is sufficient to show that (a) $trA < 1 + \det A$ (b) $\det A < 1$. First consider

(a) $trA < 1 + \det A$

$$\frac{\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \gamma_1 + \frac{\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_2 < 1 + \frac{\gamma_2\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \frac{\gamma_1\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_1\gamma_2 .$$

Further simplification using $\theta = \frac{\beta_1}{1-\gamma_1} - 1$ and $\phi = \frac{\beta_2}{1-\gamma_2} - 1$ gives

$$(1-\gamma_1)(1-\gamma_2)\frac{(1+\alpha_{12}\overline{N}_2)}{(1+\theta)} + (1-\gamma_2)(1-\gamma_1)\frac{(1+\alpha_{21}\overline{N}_1)}{(1+\phi)} < (1-\gamma_1)(1-\gamma_2) .$$

$$\frac{(1+\alpha_{12}\overline{N}_2)}{(1+\theta)} + \frac{(1+\alpha_{21}\overline{N}_1)}{(1+\phi)} < 1 \qquad .$$

Using (2.13), we have

$$\frac{\beta_1 - (1-\gamma_1)\overline{N}_1}{\beta_1} + \frac{(1-\gamma_2)(1+\alpha_{21}\overline{N}_1)}{\beta_2} < 1 ,$$

$$\beta_1\beta_2 - \beta_2(1-\gamma_1)\overline{N}_1 + \beta_1(1-\gamma_2) + \beta_1\alpha_{21}(1-\gamma_2)\overline{N}_1 < \beta_1\beta_2 ,$$

Substituting for $\overline{N}_1$ using (2.14) gives

$$-\frac{\beta_1\gamma_2(1-\alpha_{12}\alpha_{21}) - [\beta_2(1-\gamma_1) - \beta_1\alpha_{21}(1-\gamma_2)]\alpha_{12}\phi}{[\beta_2(1-\gamma_1) - \beta_1\alpha_{21}(1-\gamma_2)]} < \theta .$$

Compactly written as

$$-(B_1 - \alpha_{12}\phi) < \theta , \tag{2.26}$$

where

$$B_1 = \frac{\beta_1\gamma_2(1-\alpha_{12}\alpha_{21})}{[\beta_2(1-\gamma_1)-\beta_1\alpha_{21}(1-\gamma_2)]}.$$

Now, the second inequality of the stability conditions
(b) $\det A < 1$ requires

$$\frac{\gamma_2\beta_1(1+\alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \frac{\gamma_1\beta_2(1+\alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_1\gamma_2 < 1,$$

Similarly, further simplification using (2.13), gives

$$\gamma_1\beta_1(1-\gamma_2)^2 - [\gamma_2\beta_2(1-\gamma_1)^2 - \alpha_{21}\gamma_1\beta_1(1-\gamma_2)^2]\overline{N}_1 < 1,$$

$$-\frac{[\gamma_2\beta_2(1-\gamma_1)^2 - \alpha_{21}\gamma_1\beta_1(1-\gamma_2)^2]\alpha_{12}\phi - [\gamma_1\beta_1(1-\gamma_2)^2 - 1](1-\alpha_{12}\alpha_{21})}{[\gamma_2\beta_2(1-\gamma_1)^2 - \alpha_{21}\gamma_1\beta_1(1-\gamma_2)^2]} < \theta,$$

Compactly written as

$$-(\alpha_{12}\phi - B_2) < \theta. \tag{2.27}$$

where,

$$B_2 = \frac{-[\gamma_1\beta_1(1-\gamma_2)^2 - 1](1-\alpha_{12}\alpha_{21})}{[\gamma_2\beta_2(1-\gamma_1)^2 - \alpha_{21}\gamma_1\beta_1(1-\gamma_2)^2]}.$$

If the inequalities (2.26) and (2.27) are fulfilled, this implies that the two stability conditions are automatically satisfied. Therefore E_1 is locally stable.

2.2.8 Stability of the Steady state for a crop and one species of weed coexistence

The coexistence fixed points of a crop and a species of weed that survive the control (a weed died out due to control) is analyzed. Evaluating (2.21) at $E_2(\overline{N}_c, \overline{N}_1, 0)$, using (2.15) and (2.16) it becomes

$$D(E_2) = \begin{pmatrix} \dfrac{(1+\alpha_{c1}\overline{N}_1)}{\beta_c} & \dfrac{-\alpha_{c1}\overline{N}_c}{\beta_c} & \dfrac{-\alpha_{c2}\overline{N}_c}{\beta_c} \\ \dfrac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\theta)^2} & \dfrac{\beta_1(1+\alpha_{1c}\overline{N}_c)}{(1+\theta)^2} + \gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\theta)^2} \\ 0 & 0 & \dfrac{\beta_2}{(1+\phi)} + \gamma_2 \end{pmatrix}. \tag{2.28}$$

Equation (2.28) is a block upper rectangular matrix. However, row-operation method is used to reduce it to upper triangular matrix. Thus,

$$D(E_2)=\begin{pmatrix} \dfrac{(1+\alpha_{c1}\overline{N}_1)}{\beta_c} & \dfrac{-\alpha_{c1}\overline{N}_c)}{\beta_c} & \dfrac{-\alpha_{c2}\overline{N}_c)}{\beta_C} \\ 0 & \dfrac{(1-\gamma_1)\alpha_{1c}\overline{N}_c B+\beta_c(1+\theta\gamma_1)}{\beta_c(1+\theta)} & \dfrac{-(1-\gamma_1)\overline{N}_1\{\alpha_{1c}\alpha_{c2}\overline{N}_c+\alpha_{12}\beta_c\}}{(1+\theta)\beta_c} \\ 0 & 0 & \dfrac{1+\phi\gamma_2}{(1+\phi)} \end{pmatrix}. \qquad (2.29)$$

(2.29) is an upper triangular matrix, so, its eigenvalues are equal to the elements along the principal diagonal. For the stability it requires that

$$\lambda_1=\frac{(1+\alpha_{c1}\overline{N}_1)}{\beta_c}<1,\quad \lambda_2=\frac{(1-\gamma_1)\alpha_{1c}\overline{N}_c B+\beta_c(1+\theta\gamma_1)}{\beta_c(1+\theta)}<1,\quad \lambda_3=\frac{1+\phi\gamma_2}{(1+\phi)}<1,$$

where $B=\beta_c-\alpha_{1c}\overline{N}_1$.

Simplifying $\dfrac{1+\phi\gamma_2}{(1+\phi)}<1$ gives $\gamma_2<1$, this implies $\lambda_3<1$.

So, the stability depends on λ_1 and λ_2. Now simplify λ_1, that is $\dfrac{(1+\alpha_{c1}\overline{N}_1)}{\beta_c}<1$.

Using (2.16), we have

$$\theta\alpha_{c1}-\alpha_{c1}\alpha_{1c}(\beta_c-1)<(\beta_c-1)(1-\alpha_{c1}\alpha_{1c}).$$

Further simplification yields

$$\alpha_{c1}<\frac{(\beta_c-1)}{\theta}. \qquad (2.30)$$

It implies that $\lambda_1<1$, provided the inequality (2.30) holds.

Also, simplifying λ_2, thus

$$\frac{(1-\gamma_1)\alpha_{1c}\overline{N}_c B+\beta_c(1+\theta\gamma_1)}{\beta_c(1+\theta)}<1 \text{ implies } \alpha_{1c}\overline{N}_c B<\theta\beta_c.$$

Substituting for $\overline{N}_c$ gives $-\alpha_{1c}B(1+\alpha_{c1}\theta)<\beta_c[\theta(1-\alpha_{c1}\alpha_{1c})-\alpha_{1c}B]$,

Substituting for B then $\overline{N}_1$ gives

$$\beta_c\alpha_{1c}-\alpha_{1c}(1+\alpha_{c1}\theta)\left(\frac{\beta_c(1-\alpha_{c1}\alpha_{1c})-\theta\alpha_{1c}+\alpha_{1c}\alpha_{1c}(\beta_c-1)}{(1-\alpha_{c1}\alpha_{1c})}\right)<\beta_c[\theta(1-\alpha_{c1}\alpha_{1c})].$$

After simplification, it becomes

$$\frac{\alpha_{1c}{}^{2}(1+\alpha_{c1}\theta)(\theta+\alpha_{1c})}{[\theta(1-\alpha_{c1}\alpha_{1c})^{2}+\alpha_{1c}\alpha_{c1}(\theta+\alpha_{1c})]}<\beta_c \; . \qquad (2.31)$$

So, $\lambda_2 < 1$, provided the inequality (2.31) holds.

Similarly, evaluating (2.21) at the steady state $E_3(\overline{N}_c, 0, \overline{N}_2)$, applying row-operation, it becomes

$$D(E_3)=\begin{pmatrix} \frac{1+\alpha_{c2}\overline{N}_2}{\beta_c} & \frac{-\alpha_{c1}\overline{N}_c}{\beta_c} & \frac{-\alpha_{c2}\overline{N}_c}{\beta_c} \\ 0 & \frac{\beta_1}{(1+\theta)}+\gamma_1 & 0 \\ 0 & \frac{\alpha_{2c}\alpha_{c1}\beta_2\overline{N}_c\overline{N}_2-\alpha_{c2}\alpha_{21}\beta_2\overline{N}_2{}^{2}}{\beta_c(1-\phi)^{2}} & \frac{\gamma_2(1+\alpha_{c2}\overline{N}_2)}{\beta_c} \end{pmatrix}. \qquad (2.32)$$

The characteristic equation is

$$\left(\frac{1+\alpha_{c2}\overline{N}_2}{\beta_c}-\lambda\right)\left(\frac{\beta_1}{(1+\theta)}+\gamma_1-\lambda\right)\left(\frac{\gamma_2(1+\alpha_{c2}\overline{N}_2)}{\beta_c}-\lambda\right)=0 \; .$$

So, its eigenvalues are

$$\lambda_1=\left(\frac{1+\alpha_{c2}\overline{N}_2}{\beta_c}\right), \; \lambda_2=\left(\frac{\beta_1}{(1+\theta)}+\gamma_1\right) \text{ and } \lambda_3=\left(\frac{\gamma_2(1+\alpha_{c2}\overline{N}_2)}{\beta_c}\right).$$

For the stability it requires that

$\lambda_1=\dfrac{1+\alpha_{c2}\overline{N}_2}{\beta_c}<1$ implies $1+\alpha_{c2}\overline{N}_2<\beta_c$ and simplified using (2.18) to obtain

$$\alpha_{c2}<\frac{(\beta_c-1)}{\phi}. \qquad (2.33)$$

Also

$$\lambda_2=\frac{\beta_1}{(1+\theta)}+\gamma_1<1.$$

Since weed N_1 is extinct (due to control) $\beta_1=0$, it implies $\gamma_1<1$. Similarly

$$\lambda_3 = \frac{\gamma_2 (1+\alpha_{c2}\overline{N}_2)}{\beta_C} < 1 \text{ implies } \gamma_2 \alpha_{c2} \overline{N}_2 < (1-\gamma_2)\beta_c .$$

Employ (2.18) and simplified to obtain $\gamma_2 \phi \alpha_{c2} < \beta_c - \gamma_2$, hence

$$\alpha_{c2} < \frac{(\beta_c - \gamma_2)}{\gamma_2 \phi}. \tag{2.34}$$

So, E_3 is locally asymptotically stable provided the inequalities (2.33) and (2.34) hold. The stability results are summarized in Table 2.2.

Table 2.2. Stability conditions of the steady states for two plant species interaction

Steady-state	Condition	Stability result
E_0	$\beta_c > 1, \beta_2 + \gamma_2 > 1, \beta_1 + \gamma_1 > 1$	unstable
E_1	$\frac{1}{\beta_c} < 1$, $-(B_1 - \alpha_{12}\phi) < \theta$, $-(\alpha_{12}\phi - B_2) < \theta$	stable
E_2	$\gamma_2 < 1$, $\alpha_{c1} < \frac{(\beta_c - 1)}{\theta}$, $\frac{\alpha_{1c}{}^2 (1+\alpha_{c1}\theta)(\theta+\alpha_{1c})}{[\theta(1-\alpha_{c1}\alpha_{1c})^2 + \alpha_{1c}\alpha_{c1}(\theta+\alpha_{1c})]} < \beta_c$	stable
E_3	$\gamma_1 < 1,$ $\alpha_{c2} < \frac{(\beta_c - 1)}{\phi}$, $\alpha_{c2} < \frac{(\beta_c - \gamma_2)}{\gamma_2 \phi}$	stable

2.2.9 Stability of a crop and two-species of weed co-existence steady state

The linearization of system (2.6) – (2.8) about the steady state, $E(\overline{N}_c, \overline{N}_1, \overline{N}_2)$ as given in (2.20) the partial derivative matrix (2.21) after simplification yields

$$D(E_4) = \begin{pmatrix} \frac{(1+\alpha_{c1}\overline{N}_1 + \alpha_{c2}\overline{N}_2)}{\beta_c} & \frac{-\alpha_{c1}\overline{N}_c}{\beta_c} & \frac{-\alpha_{c2}\overline{N}_c}{\beta_c} \\ \frac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\theta)^2} & \frac{\beta_1(1+\alpha_{1c}\overline{N}_c + \alpha_{12}\overline{N}_2)}{(1+\theta)^2} + \gamma_1 & \frac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\theta)^2} \\ \frac{-\alpha_{2c}\beta_2\overline{N}_2}{(1+\phi)^2} & \frac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\phi)^2} & \frac{\beta_2(1+\alpha_{2c}\overline{N}_c + \alpha_{21}\overline{N}_1)}{(1+\phi)^2} + \gamma_2 \end{pmatrix}.$$

For easy analysis it is expressed as

$$D(E_4)=\begin{pmatrix} \dfrac{X_c}{\beta_c} & \dfrac{-\alpha_{c1}\overline{N}_c}{\beta_c} & \dfrac{-\alpha_{c2}\overline{N}_c}{\beta_c} \\ \dfrac{-\alpha_{1c}\beta_1\overline{N}_1}{(1+\theta)^2} & \dfrac{\beta_1 X_1}{(1+\theta)^2}+\gamma_1 & \dfrac{-\alpha_{12}\beta_1\overline{N}_1}{(1+\theta)^2} \\ \dfrac{-\alpha_{2c}\beta_2\overline{N}_2}{(1+\phi)^2} & \dfrac{-\alpha_{21}\beta_2\overline{N}_2}{(1+\phi)^2} & \dfrac{\beta_2 X_2}{(1+\phi)^2}+\gamma_2 \end{pmatrix}, \tag{2.35}$$

where

$X_c=1+\alpha_{c1}\overline{N}_1+\alpha_{c2}\overline{N}_2$, $X_1=1+\alpha_{1c}\overline{N}_c+\alpha_{12}\overline{N}_2$ and $X_2=1+\alpha_{2c}\overline{N}_c+\alpha_{21}\overline{N}_1$.

The characteristic equation of matrix (2.35) is given by

$$f(\lambda)=\left(\frac{X_c}{\beta_c}-\lambda\right)\left[\left(\frac{\beta_1X_1+(1+\theta)^2\gamma_1}{(1+\theta)^2}-\lambda\right)\left(\frac{\beta_2X_2+(1+\phi)^2\gamma_2}{(1+\phi)^2}-\lambda\right)-\frac{\alpha_{12}\alpha_{21}\beta_1\beta_2\overline{N}_1\overline{N}_2}{(1+\theta)^2(1+\phi)^2}\right]$$

$$-\frac{\alpha_{1c}B_1\overline{N}_1}{\theta^2}\left[\frac{\alpha_{c1}\overline{N}_c}{\beta_c}\left(\frac{B_2X_2+(1+\phi)^2\gamma_2}{(1+\phi)^2}-\lambda\right)+\frac{\alpha_{c2}\alpha_{21}B_2\overline{N}_c\overline{N}_2}{\beta_c(1+\phi)^2}\right]$$
$$-\frac{\alpha_{2c}\beta_2\overline{N}_2}{(1+\phi)^2}\left[\frac{\alpha_{c1}\alpha_{12}\beta_1\overline{N}_c\overline{N}_1}{\beta_c(1+\theta)^2}+\frac{\alpha_{c2}\overline{N}_c}{\beta_c}\left(\frac{\beta_1X_1+(1+\theta)^2\gamma_1}{(1+\theta)^2}-\lambda\right)\right].$$

It is simplified to obtain

$$\begin{aligned} f(\lambda)&=\lambda^3+\\ &\frac{X_c(\beta_1X_1+\theta^2\gamma_1)(\beta_2X_2+\phi^2\gamma_2)+X_c\phi^2\theta^2-[(\beta_2X_2+\phi^2\gamma_2)\theta^2+(\beta_1X_1+\theta^2\gamma_1)\phi^2])}{\beta_c\theta^2\phi^2}\lambda^2\\ &-\frac{\alpha_{c1}\alpha_{1c}\beta_1\overline{N}_c\overline{N}_1+\alpha_{2c}\alpha_{c2}\beta_2\overline{N}_c\overline{N}_2+[1-X_c(\theta^2+\phi^2)](\beta_1X_1+\theta^2\gamma_1)(\beta_2X_2+\phi^2\gamma_2)}{\beta_c\theta^2\phi^2}\lambda\\ &+\frac{\alpha_{c1}\alpha_{1c}\beta_1\overline{N}_c\overline{N}_1(\beta_2X_2+\phi^2\gamma_2)+\alpha_{2c}\alpha_{c2}\beta_2\overline{N}_c\overline{N}_2(\beta_1X_1+\theta^2\gamma_1)}{\beta_c\theta^2\phi^2}\\ &\frac{-(X_c+\beta_c)\alpha_{12}\alpha_{21}\beta_1\beta_2\overline{N}_1\overline{N}_2-(\alpha_{c1}\alpha_{2c}\alpha_{12}+\alpha_{c2}\alpha_{1c}\alpha_{21})\beta_1\beta_2\overline{N}_c\overline{N}_1\overline{N}_2}{\beta_c\theta^2\phi^2}, \end{aligned}$$

Then re-expressed as

$$f(\lambda)=\lambda^3+\frac{X_c(A_1A_2+\phi^2\theta^2)-(A_2\theta^2+A_1\phi^2)}{\beta_c\theta^2\phi^2}\lambda^2-\frac{A_3+A_4+[1-X_c(\theta^2+\phi^2)]A_1A_2}{\beta_c\theta^2\phi^2}\lambda$$
$$+\frac{A_2A_3+A_1A_4-(X_c+\beta_c)A_5-A_6}{\beta_c\theta^2\phi^2}\ , \tag{2.36}$$

where

$$A_1 = (B_1 X_1 + \theta^2 \gamma_1),\quad A_2 = (B_2 X_2 + \phi^2 \gamma_2),\quad A_3 = \alpha_{c1}\alpha_{1c} B_1 \overline{N}_c \overline{N}_1 ,\quad A_4 = \alpha_{2c}\alpha_{c2} B_2 \overline{N}_c \overline{N}_2$$
$$A_5 = \alpha_{12}\alpha_{21} B_1 B_2 \overline{N}_1 \overline{N}_2 ,\quad A_6 = (\alpha_{c1}\alpha_{2c}\alpha_{12} + \alpha_{c2}\alpha_{1c}\alpha_{21}) B_1 B_2 \overline{N}_c \overline{N}_1 \overline{N}_2 .$$
,

In a more compact form (2.36) becomes

$$f(\lambda) = \lambda^3 + a_1\lambda^2 - a_2\lambda + a_3 = 0, \tag{2.37}$$

where

$$a_1 = \frac{X_c(A_1A_2 + \phi^2\theta^2) - (A_2\theta^2 + A_1\phi^2)}{\beta_c\theta^2\phi^2} \quad , a_2 = \frac{A_3 + A_4 + [1 - X_c(\theta^2 + \phi^2)]A_1A_2}{\beta_c\theta^2\phi^2},$$

$$a_3 = \frac{A_2A_3 + A_1A_4 - (X_c + \beta_c)A_5 - A_6}{\beta_c\theta^2\phi^2} .$$

In order to determine the stability of the positive coexistence steady state E_4, we discuss the roots of the equation (2.37), which are the eigenvalues of matrix D(E_4) by applying the Jury criteria for local stability test.

The necessary and sufficient conditions for the characteristic polynomial $f(\lambda)$ to have roots inside the unit circle (that is all have magnitude less than 1) are given as;

$f(1) > 0,\quad (-1)^3 f(-1) > 0,$ and $|a_3| < 1,$ the Jury's Criteria [12]. So, applying these to (2.37);

$$f(1) = 1 + a_1 - a_2 + a_3 > 0 .$$
$$(-1)^3 f(-1) = 1 - (a_1 + a_2 + a_3) > 0 .$$

Provided $(a_1 + a_2 + a_3) < 1$. Furthermore, the constant term satisfies

$$|a_3| = \left| \frac{A_2A_3 + A_1A_4 - (X_c + \beta_c)A_5 - A_6}{\beta_c\theta^2\phi^2} \right| < 1,$$
$$\frac{A_2A_3 + A_1A_4 - X_C A_5 - A_6}{(A_5 + \theta^2\phi^2)} < \beta_C .$$

If the conditions for the Jury criterion are satisfied, it implies that the three roots of the equation (2.37) satisfy $\lambda_1 < 1$, $\lambda_2 < 1$, and $\lambda_3 < 1$. The linearization theory implies that the positive steady state E_4 $(\overline{N}_c, \overline{N}_1, \overline{N}_2)$ is locally asymptotically stable if the above inequalities hold. So, the coexistence steady state E_4 of our model equations is locally asymptotically stable and the crop's growth may stagnate, but survive in the mixture of the two species of weeds. However, if the effective control measure is applied to eradicate the two weeds species the crop will grow at its optimum capacity.

3 Results and Discussion

In this section, the competition models are studied numerically by chosen parameter values to test (explain) the analytical results obtained. The values were carefully chosen to satisfy the necessary conditions for stability of the steady-states solutions. The values of the parameters for the graphical profiles are selected mathematically to let the model demonstrate the required dynamics.

3.1 Two-Weed Species Coexistence

The parameter values we employ were selected to satisfy the necessary conditions for stability of the steady-states solutions under consideration. Therefore, the parameter values selected for numerical explanation of the coexistence fixed points of the two-species (N_c , N_1 , 0) are $\beta_c = 2$, $\theta = 1$, $\phi = 1$, $\alpha_{1c} = 0.8$, $\alpha_{2c} = 1$, $\alpha_{c1} = 0.5$, $\alpha_{c2} = 0.9$, $\alpha_{12} = 1.4$, and $\alpha_{21} = 1.1$.

These satisfy the algebraic criteria for the existence of the non-zero fixed pints, E_2 . So the numerical result of the steady-states solution converges to $(N_c(t), N_1(t), 0) = (0.83, 0.33, 0)$.This gives the coexistence steady state of one crop and one species of weed. (i.e the second weed extinct). Besides, the numerical result shows that the steady state is locally asymptotically stable since the parameter values satisfy the stability conditions in Table 2.1.

3.2 Crop and Two-Weed Species Coexistence

The parameter values employed for numerical explanation of the coexistence fixed points were selected to satisfy the algebraic criteria for the existence of the non-zero steady state, E_4 . Therefore, the parameter values are the following;

$\beta_c = 2$, $\theta = 1$, $\phi = 1$, $\alpha_{1c} = 1.6$, $\alpha_{2c} = 0.6$, $\alpha_{c1} = 0.5$, $\alpha_{c2} = 1.4$, $\alpha_{12} = 0.6$, and $\alpha_{21} = 1.5$.

Using these values the steady- state solutions converges to $E_4 = (\overline{N}_c, \overline{N}_1, \overline{N}_2) = (1.00, 0.27, 0.80)$. That is, the crop would have its maximum population at the steady state.

We obtain the characteristic equation

$$f(\lambda) = \lambda^3 + 11.4\lambda^2 - 9\lambda + 1.7 = 0 . \tag{3.1}$$

Table 3.1. Jury's stability criterion for weed-crop competition model

	Condition	System	Stability result
1	$f(1) > 0$	$f(1) = 1.7 > 0$	Satisfied
2	$(-1)^3 f(-1) > 0$	$(-1)^3 f(-1) = -17.7 < 0$	Not satisfied
3	$\lvert a_3 \rvert < 1$	$\lvert 1.7 \rvert = 1.7 > 1$	Not satisfied

Following the stability results obtained in Table 3.1, the three roots of the equation (3.1) are not within the disk (not less than 1). Hence, the weed-crop competition model is not locally asymptotically stable for chosen values at the positive steady state $(\overline{N}_c, \overline{N}_1, \overline{N}_2)$. This implies that, the crop may not grow and survive (stagnated) in the mixture of the two species of weeds if the weeds densities are not controlled.

3.3 Graphical Profile for Crop and Two-Species of Weed Coexistence

In this section, the graphical profile for a crop and two-species of weed interaction (co-existence) was obtained using Mathematica.

Fig. 3.1 depicts the dynamics of weed-crop competition model equations (2.6) - (2.8) using the time intervals of 1 week. The graphical profiles show that the density of the two weeds continued to be on the increase during the growing season as long as there are enough resources to promote their growth, but approach steady densities after 30 weeks. Besides, it supported the co-existence analysis result obtained in section 2.2.9 that, the growth of the crop may be stagnated if the weeds densities are not controlled. The 1st (first) and 2nd (second) weeds' densities growth rates dominate that of the crop, while that of the 1st weed dominates that of the 2nd weed (i.e one overgrown the other).

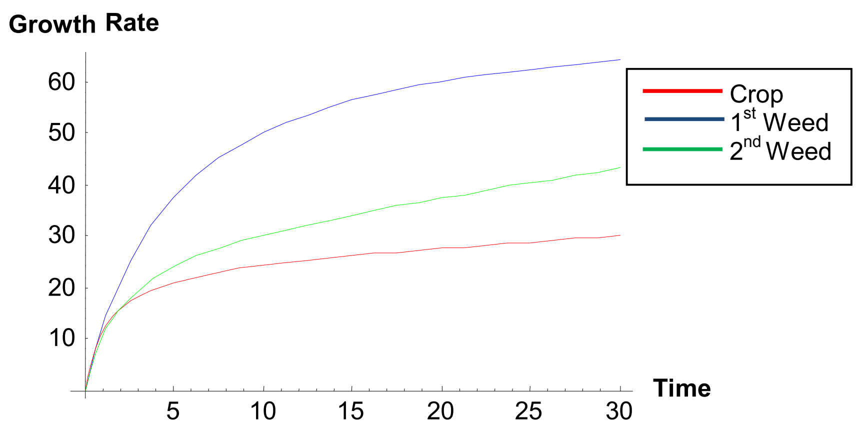

Fig. 3.1. Rate of growth of crop and two-species of weed coexistence

4 Conclusions

The main idea studied in this paper is the formulation of a model to investigate the weed-crop competition dynamics for the above ground established weeds densities. The model equations were analyzed for stability based on variational principles. From the analysis, the following findings / conclusion can be drawn.

i. The extinction steady-state point E_0 (0,0,0) is not stable without control.
ii. The coexistence steady state of weed-crop competition model is locally asymptotically stable.
iii. Experimental work showed that the positive steady state solution of weed-crop competition model is not stable based on the chosen values. We conclude that the crop may not grow and survive in the mixture of the two species of weeds at certain level of weeds infestation. That is its growth may be stagnated as depicts in the model's graphical profile.

Based on the results, the crop's growth may be stagnated, but survive in the mixture of the two species of weeds. However, application of effective control measure to eradicate the two weeds species will improve crop yield at its optimum capacity for sustainable food production.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Akobundu IO. Weed science in the tropics: Principles and practice. International Institute of Tropical Agriculture, Ibadan, A Wiley-Inter-science publication. John Wiley & Sons Ltd, Great Britain. 1987;45.

[2] Liang X, Ru R, Peng X. Research progress of vector-borne plant disease. Biological Eng. Prog. 2001;21:11-17.

[3] Shi R, Zhao H, Tang S. Global dynamic analysis of a vector-borne plant disease modal. Advance in Difference Equation. 2014;59.

[4] Allen LJS, Allen EJ, Ponweera S. A mathematical model for weed dispersal and control. Bulletin Weed Mathematical Biology. 1996;58(5):815-834. Elsevier Science Inc.

[5] Alsharawi Z, Rhouma MBH. The discrete beverton-holt model with periodic harvesting in a periodically fluctuating environment. Advances in Difference Equation; 2010. Article ID 215875.

[6] David NA. Mathematical models for plant competition and dispersal. (Unpublished M.Sc Thesis), The Graduate Faculty of Texas Technology University; 1997.

[7] Sacker JR. Global stability in a multi-species periodic Leslie-Gower model. Journal of Biological Dynamics. Taylor & Francis; 2010. DOI: 10.1080/1751375YY.
Available: www.tandf.co.uk/journals.

[8] Wu D, Zhang H. Bifurcation analysis of a two-species competitive discrete model of plankton allelopathy. Advances in Difference Equations. 2014:70.
Available: www.advancesindifferenceequations.com/contsnt/2014/1/70.

[9] Nasir MO, Akinwande NI, Kolo MGM, Mohammed J. A Discrete-Time Mathematical Model for Homogeneous Population Density Dynamics of Single Weed Species. Journal of Mathematical Theory and Modeling; 2014. (Accepted for publication).
Available: www.iiste.org.

[10] Agarwal RP. Difference Equations and Inequalities; Theory, Methods, and Applications. 2nd ed., Marcel Dekker. Inc New York, NY 10016; 2000.

[11] Elaydi SN. Discrete chaos with applications in science and engineering. Chapman & Hall/CRC, Taylor & Francis Group. London; 2008.

[12] Xiaoliang L, Chenqi M, Wei N, Dongming W. Stability analysis for discrete biological models using algebraic methods. Journal of Mathematics in Computer Science. 2011;5(3):247-262. DOI: 10.1007/S11786-011.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=1032&id=6&aid=8697