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Quartic Cyclic Homogeneous Polynomial Inequalities of Three Nonnegative Variables

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Abstract

We present and prove a set of necessary and sufficient conditions that the inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real variables x, y, z, where $f_4(x, y, z)$ is a cyclic homogeneous polynomial of degree four which satisfies $f_4(1, 1, 1) = 0$.

Keywords: Cyclic homogeneous inequality; quartic polynomial; nonnegative variables; necessary and sufficient conditions.

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1 Introduction

A quartic cyclic homogeneous polynomial of three variables has the form

$$f_4(x, y, z) = x^4 + y^4 + z^4 + A(x^2y^2 + y^2z^2 + z^2x^2) + Bxyz(x + y + z)$$
$$+C(x^3y + y^3z + z^3x) + D(xy^3 + yz^3 + zx^3),$$

where A, B, C, D are real constants.

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In [1], V. Cirtoaje presented and proved that

$$3(1+A) \geq C^2 + CD + D^2$$

is a necessary and sufficient condition to have $f_4(x, y, z) \geq 0$ for all real x, y, z in the particular case $f_4(1, 1, 1) = 0$.

In [2], we obtained two set of necessary and sufficient conditions to have $f_4(x, y, z) \geq 0$ for all real x, y, z in the general case $f_4(1, 1, 1) \geq 0$. These conditions are stated in Theorem 1.1 and Theorem 1.2.

Theorem 1.1. *The cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if and only if*

$$f_4(t+k, k+1, kt+1) \geq 0$$

for all real t, where $k \in [0, 1]$ is a root of the equation

$$(C-D)k^3 + (2A - B - C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D = 0.$$

Theorem 1.2. *The cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if and only if $g_4(t) \geq 0$ for all $t \geq 0$, where*

$$g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C-D)^2 + E^2}, \quad E = 8 - 4A + 2B - C - D.$$

The following theorem in [3] expresses some strong sufficient conditions that the inequality $f_4(x, y, z) \geq 0$ holds for all real x, y, z.

Theorem 1.3. *Let*

$$G = \sqrt{1 + A + B + C + D},$$
$$H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if the following two conditions are satisfied:

(a) $1 + A + B + C + D \geq 0$;

(b) *there exists a real number $t \in (-\sqrt{3}, \sqrt{3})$ such that $f(t) \geq 0$, where*

$$f(t) = 2Gt^3 - (6 + 2A + B + 3C + 3D)t^2 + 2(1 + C + D)Gt + H.$$

In [4], we found some sharp sufficient conditions that the inequality $f_4(x, y, z) \geq 0$ holds for all $x, y, z \geq 0$, which are stated in Theorem 1.4.

Theorem 1.4. *The inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z if*

$$1 + A + B + C + D \geq 0$$

and one of the following two conditions is fulfilled:

(a) $3(1 + A) \geq C^2 + CD + D^2$;

(b) $3(1 + A) < C^2 + CD + D^2$ *, and there is $t \geq 0$ such that*

$$(C + 2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)}.$$

In addition, we have conjectured that for $1 + A + B + C + D = 0$, the conditions (a) and (b) in Theorem 1.4 are necessary and sufficient to have $f_4(x, y, z) \geq 0$ for all $x, y, z \geq 0$. The main objective of this paper is to show that this conjecture is true. Some related results are also given in [5], [6] and [7].

2 Main Results

The main result is given by the theorem below, which gives a set of necessary and sufficient conditions to have $f_4(x,y,z) \geq 0$ for all $x,y,z \geq 0$ in the most usual case $f_4(1,1,1) = 0$.

Theorem 2.1. *For $f_4(1,1,1) = 0$, the inequality $f_4(x,y,z) \geq 0$ holds for all nonnegative real numbers x,y,z if and only if one of the following two conditions is satisfied:*

(a) $3(1+A) \geq C^2 + CD + D^2$;

(b) $3(1+A) < C^2 + CD + D^2$ *, and there exists $t_0 \geq 0$ such that*

$$F_4(t_0) = (2C+D)t_0^2 + 6t_0 + 2D + C - 2\sqrt{(t_0^4 + t_0^2 + 1)(C^2 + CD + D^2 - 3(1+A))} \geq 0.$$

Consider now the more general case where $f_4(1,1,1) \geq 0$. Applying Theorem 2.1 to the function

$$g_4(x,y,z) = f_4(x,y,z) - (1 + A + B + C + D)xyz\sum x,$$

which satisfies $g_4(1,1,1) = 0$, we get the following corollary.

Corollary 2.2. *The inequality $f_4(x,y,z) \geq 0$ holds for all nonnegative real numbers x,y,z if one of the following two conditions is satisfied:*

(a) $3(1+A) \geq C^2 + CD + D^2$;

(b) $3(1+A) < C^2 + CD + D^2$ *, and there exists $t_0 \geq 0$ such that*

$$F_4(t_0) = (2C+D)t_0^2 + 6t_0 + 2D + C - 2\sqrt{(t_0^4 + t_0^2 + 1)(C^2 + CD + D^2 - 3(1+A))} \geq 0.$$

To prove Theorem 2.1, we need three lemmas.

Lemma 2.3. *Let*

$$S = \sum x^2y^2 - \sum x^2yz, \quad U = \frac{\sum x^3y - \sum x^2yz}{S}, \quad V = \frac{\sum xy^3 - \sum x^2yz}{S}.$$

If $x,y,z \geq 0$, then

$$U > 0, \quad V > 0, \quad UV = 1 + \frac{xyz(x+y+z)(x^2+y^2+z^2-xy-yz-zx)^2}{S^2} \geq 1.$$

In addition, for $f_4(1,1,1) = 0$, the inequality

$$f_4(x,y,z) \geq 0$$

holds for all real x,y,z if and only if

$$F(U,V) \geq 0,$$

where

$$\begin{aligned} 4F(U,V) &= 4(U^2 - UV + V^2 + 1 + A + CU + DV) \\ &= (U+V+C+D)^2 + 3\left(U - V + \frac{C-D}{3}\right)^2 + \frac{4}{3}(3 + 3A - C^2 - CD - D^2). \end{aligned}$$

Lemma 2.4. *If t_0 is a real root of the equation*

$$2t^4 + Dt^3 - Ct - 2 = 0,$$

then

$$\left(\frac{1}{t_0} + t_0 + C + D\right)^2 + 3\left(\frac{1}{t_0} - t_0 + \frac{C-D}{3}\right)^2 = \frac{[(2C+D)t_0^2 + 6t_0 + C + 2D]^2}{3(t_0^4 + t_0^2 + 1)}.$$

Lemma 2.5. *Let t_0 be a real root of the equation*

$$2t^4 + Dt^3 - Ct - 2 = 0.$$

If

$$3(1+A) < C^2 + CD + D^2,$$

$f_4(1,1,1) = 0$ *and* $f_4(x,y,z) \geq 0$ *for all* $x, y, z \geq 0$*, then*

$$(2C+D)t_0^2 + 6t_0 + C + 2D \geq 0.$$

3 Proof of Lemmas 2.3, 2.4 and 2.5

Proof of Lemma 2.3. From

$$2S = x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2,$$

it follows that $S \geq 0$. In addition, $S = 0$ when $x = y = z$, and also when $y = z = 0$ (or any cyclic permutation). For $x, y, z \geq 0$, by the Cauchy-Schwarz inequality, we have

$$(z+x+y)(x^3y + y^3z + z^3x) \geq xyz(x+y+z)^2,$$

hence

$$x^3y + y^3z + z^3x \geq xyz(x+y+z),$$

with equality for $x = y = z$. From this inequality and $S \geq 0$, it follows that $U > 0$. Similarly, we can show that $V > 0$. To complete the proof, we use the identity

$$\frac{f_4(x,y,z)}{S} = F(U,V),$$

which is valid for all real x, y, z such that $S \neq 0$. Consider now the case $S = 0$. If $x = y = z$, then

$$f_4(x,y,z) = x^4 f_4(1,1,1) = 0.$$

Also, if $y = z = 0$, we have

$$f_4(x,y,z) = x^4 \geq 0.$$

Remark 3.1. Consider the case where $f_4(1,1,1) = 0$ and $3(1+A) = C^2 + CD + D^2$. In order to study when the equality $f_4(x,y,z) = 0$ occurs (for other cases than $x = y = z$), assume that

$$p = x+y+z,\ q = xy+yz+zx,\ r = xyz.$$

We have the following two identities

$$UV - 1 = \frac{pr(p^2-3q)^2}{(q^2-3pr)^2},$$

$$U + V - 1 = \frac{q(p^2 - 3q)}{q^2 - 3pr}.$$

Without loss of generality, assume that $p = x + y + z = 3$. After some calculations, we get

$$\begin{cases} p = x + y + z = 3, \\ q = xy + yz + zx = \frac{9(U+V-1)}{U^2+V^2-UV+U+V+1}, \\ r = xyz = \frac{27(UV-1)}{(U^2+V^2-UV+U+V+1)^2}. \end{cases} \tag{3.1}$$

Since the equality $f_4(x, y, z) = 0$ holds for $U + V = -C - D$ and $U - V = (-C + D)/3$ (Lemma 2.3), we get the equality conditions

$$\begin{cases} p = x + y + z = 3, \\ q = xy + yz + zx = \frac{-108(C+D+1)}{(C-D)^2+3(C+D-2)^2}, \\ r = xyz = \frac{108(9(C+D)^2-(C-D)^2-36)}{((C-D)^2+3(C+D-2)^2)^2}, \end{cases} \tag{3.2}$$

which are the same as the ones in [2].

Remark 3.2. Let $f_4(x, y, z)$ be a fourth degree cyclic homogeneous polynomial such that $f_4(1, 1, 1) = 0$ and $f_4(x, y, z) \geq 0$ for all real numbers x, y, z. The inequality $f_4(x, y, z) \geq 0$ becomes an equality when $x = y = z$, and also when x, y, z satisfy

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0$$

and are proportional to the roots w_1, w_2 and w_3 of the polynomial equation

$$w^3 - 3w^2 + qw - r = 0.$$

Proof of Lemma 2.4. Denote

$$G = \frac{1}{t_0} + t_0 + C + D, \quad H = \frac{1}{t_0} - t_0 + \frac{C - D}{3}.$$

We need to show that $X = Y$, where

$$X = (G^2 + 3H^2)[(3(t_0^2 + 1)^2 + (t_0^2 - 1)^2], \quad Y = \frac{4}{3}[(2C + D)t_0^2 + 6t_0 + C + 2D]^2.$$

Since

$$\begin{aligned} X &= [G(t_0^2 - 1) - 3H(t_0^2 + 1)]^2 + 3[G(t_0^2 + 1) + H(t_0^2 - 1)]^2 \\ &= \frac{2}{t_0}(2t_0^4 + Dt_0^3 - Ct_0 - 2)^2 + 3[G(t_0^2 + 1) + H(t_0^2 - 1)]^2 \\ &= 3[G(t_0^2 + 1) + H(t_0^2 - 1)]^2, \end{aligned}$$

the desired equality becomes

$$[G(t_0^2 + 1) + H(t_0^2 - 1)]^2 = \frac{4}{9}[(2C + D)t_0^2 + 6t_0 + C + 2D]^2.$$

This is true because of

$$3[G(t_0^2 + 1) + H(t_0^2 - 1)] = 2[(2C + D)t_0^2 + 6t_0 + C + 2D].$$

Proof of Lemma 2.5. Denote

$$a = \frac{2C + D}{3}, \quad b = \frac{C + 2D}{3}, \quad g(t) = at^2 + 2t + b.$$

We need to show that $g(t_0) \geq 0$.

We will show first that there exists $t \geq 0$ such that $g(t) \geq 0$. For the sake of contradiction, assume that $g(t) < 0$ for all $t \geq 0$. From

$$g(0) = b$$

and

$$\lim_{t\to\infty} \frac{g(t)}{t^2} = a,$$

we get

$$a < 0, \quad b < 0.$$

In addition, from $g\left(\sqrt{\frac{b}{a}}\right) < 0$, we get

$$ab > 1.$$

Choosing x, y, z such that $U + V = -C - D$ and $U - V = (D - C)/3$, that is

$$U = \frac{-(2C + D)}{3} = -a > 0, \quad V = \frac{-(C + 2D)}{3} = -b > 0,$$

from Lemma 2.3 we get

$$F(U, V) = \frac{3 + 3A - C^2 - CD - D^2}{3} < 0,$$

which contradicts the hypothesis that $f_4(x, y, z) \geq 0$ for all $x, y, z \geq 0$. Therefore, there exists $t \geq 0$ such that $g(t) \geq 0$.

Since $C = 2a - b$ and $D = -a + 2b$, we can rewrite the hypothesis $2t_0^4 + Dt_0^3 - Ct_0 - 2 = 0$ in the form

$$a(t_0^3 + 2t_0) + 2 = b(2t_0^3 + t_0) + 2t_0^4, \quad t_0 > 0.$$

Using this relation gives

$$g(t_0) = \frac{2(t_0^4 + t_0^2 + 1)(at_0 + 1)}{2t_0^3 + t_0} = \frac{2(t_0^4 + t_0^2 + 1)(b + t_0)}{t_0^2 + 2},$$

from which it follows that $g(t_0) \geq 0$ for $a \geq -1/t_0$ and also for $b \geq -t_0$. To complete the proof it suffices to show that the remaining case (where $a < -1/t_0$ and $b < -t_0$) is not possible. Indeed, if $a < -1/t_0$ and $b < -t_0$, then for $t = 0$ we have $g(0) = b < 0$, and for $t > 0$ we have

$$g(t) \leq -2t\sqrt{ab} + 2t < -2t + 2t = 0.$$

This is a contradiction, because there exists $t \geq 0$ such that $g(t) \geq 0$.

4 Proof of Theorem 2.1

Sufficiency. By Lemma 2.3, it suffices to show that $F(U, V) \geq 0$.

Case (a): $3(1 + A) \geq C^2 + CD + D^2$. We have

$$\begin{aligned} 4F(U, V) &= (U + V + C + D)^2 + 3\left(U - V + \frac{C - D}{3}\right)^2 + \frac{4}{3}(3 + 3A - C^2 - CD - D^2) \\ &\geq \frac{4}{3}(3 + 3A - C^2 - CD - D^2) \geq 0. \end{aligned}$$

Case (b): $C^2 + CD + D^2 > 3(1 + A)$. Write the inequality $F(U, V) \geq 0$ in the form

$$(U + V + C + D)^2 + 3\left(U - V + \frac{C - D}{3}\right)^2 \geq \frac{4}{3}(C^2 + CD + D^2 - 3 - 3A).$$

For any $t \geq 0$, by the Cauchy-Schwarz inequality, we have

$$(U+V+C+D)^2 + 3\left(U - V + \frac{C-D}{3}\right)^2 \geq \frac{3M^2}{3(t^2+1)^2+(t^2-1)^2} = \frac{3M^2}{4(t^4+t^2+1)},$$

where

$$\begin{aligned} M &= (t^2+1)(U+V+C+D) + (t^2-1)\left(U - V + \frac{C-D}{3}\right) \\ &= \frac{2}{3}[(2C+D)t^2 + 3(Ut^2+V) + C + 2D]. \end{aligned}$$

Thus, we only need to show that

$$\frac{[(2C+D)t^2+3(Ut^2+V)+C+2D]^2}{t^4+t^2+1} \geq 4(C^2+CD+D^2-3-3A).$$

This is true if

$$\frac{(2C+D)t^2+3(Ut^2+V)+C+2D}{\sqrt{t^4+t^2+1}} \geq 2\sqrt{C^2+CD+D^2-3-3A}.$$

Since

$$Ut^2 + V \geq 2t\sqrt{UV} \geq 2t,$$

it suffices to prove that

$$(2C+D)t^2 + 6t + C + 2D \geq 2\sqrt{(t^4+t^2+1)(C^2+CD+D^2-3-3A)},$$

which is true by hypothesis.

Necessity. Let t_0 be a positive root of the equation

$$2t^4 + Dt^3 - Ct - 2 = 0.$$

It suffices to consider the case $C^2 + CD + D^2 > 3(1+A)$, and to show that if $f_4(1,1,1) = 0$ and $f_4(x,y,z) \geq 0$ for all $x, y, z \geq 0$, then

$$(2C+D)t_0^2 + 6t_0 + C + 2D \geq 2\sqrt{(t_0^4+t_0^2+1)(C^2+CD+D^2-3-3A)}.$$

By Lemma 2.4 and Lemma 2.5, we have

$$\sqrt{\left(\frac{1}{t_0} + t_0 + C + D\right)^2 + 3\left(\frac{1}{t_0} - t_0 + \frac{C-D}{3}\right)^2} = \frac{(2C+D)t_0^2 + 6t_0 + C + 2D}{\sqrt{3(t_0^4+t_0^2+1)}}.$$

Based on this result, using Lemma 2.3 for $x = 1$, $y = t_0$ and $z = 0$ yields

$$U = t_0, \qquad V = 1/t_0$$

and

$$\begin{aligned} 4F(U,V) &= \left(t_0 + \frac{1}{t_0} + C + D\right)^2 + 3\left(t_0 - \frac{1}{t_0} + \frac{C-D}{3}\right)^2 + \frac{4}{3}(3+3A-C^2-CD-D^2) \\ &= \frac{[(2C+D)t_0^2+6t_0+C+2D]^2}{3(t_0^4+t_0^2+1)} - \frac{4}{3}(C^2+CD+D^2-3-3A). \end{aligned}$$

Since the hypothesis $f_4(x,y,z) \geq 0$ for all $x,y,z \geq 0$ involves $F(U,V) \geq 0$, we get

$$\frac{[(2C+D)t_0)^2 + 6t_0 + C + 2D]^2}{3(t_0^4 + t_0^2 + 1)} \geq \frac{4}{3}(C^2 + CD + D^2 - 3 - 3A).$$

In addition, since

$$(2C+D)t^2 + 6t + C + 2D \geq 0$$

(by Lemma 2.5), we can rewrite this inequality as

$$(2C+D)t_0^2 + 6t_0 + C + 2D \geq 2\sqrt{(t_0^4 + t_0^2 + 1)(C^2 + CD + D^2 - 3 - 3A)},$$

which is just the desired inequality.

5 Conclusion

In [4], we presented and proved Theorem 1.4, which states some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative real variables and, for the most usual case $f_4(1,1,1) = 0$, we conjectured that the sufficient conditions in Theorem 1.4 are also necessary to have $f_4(x,y,z)$ for all $x,y,z \geq 0$. In this paper, we have proved that this conjecture is true.

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Competing Interests

The author declares that no competing interests exist.

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