British Journal of Mathematics & Computer Science
8(3): 189-219, 2015, Article no.BJMCS.2015.155
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

V_4- Magic Labelings of Some Wheel Related Graphs

P. T. Vandana[1] and V. Anil Kumar[1*]

[1] *Department of Mathematics, University of Calicut, Malappuram, Kerala- 673 635, India.*

Article Information
DOI: 10.9734/BJMCS/2015/16439
Editor(s):
(1) Doina Bein, Applied Research Laboratory, The Pennsylvania State University, USA.
Reviewers:
(1) Anonymous, Canada.
(2) Krasimir Yordzhev, South-West University, Faculty of Mathematics and Natural Sciences, Blagoevgrad, Bulgaria.
(3) Anonymous, USA.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=1032&id=6&aid=8602

Original Research Article

Received: 02 February 2015
Accepted: 04 March 2015
Published: 28 March 2015

Abstract

The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be V_4 magic if there exists a labeling $\ell : E(G) \to V_4 \setminus \{0\}$ such that the induced vertex labeling $\ell^+ : V(G) \to V_4$ defined by

$$\ell^+(u) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map. If this constant is equal to a, we say that ℓ is an a-sum V_4 magic labeling of G. Any graph that admits an a- sum V_4 magic labeling is called an a- magic V_4 graph. When this constant is 0 we call G a zero-sum V_4-magic graph. We divide the class of V_4 magic graphs into the following three categories:

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

In this paper, we identify some cycle related graphs which belong to the above categories.

Keywords: Klein 4-group; V_4- magic graph; Wheel graph.

2010 Mathematics Subject Classification: 05C78, 05C25

**Corresponding author: E-mail: anil@uoc.ac.in*

1 Introduction

In this paper all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not directly defined in this paper, we refer readers to [1]. For an abelian group A, written additively, any mapping $\ell : E(G) \to A \setminus \{0\}$ is called a labeling, where 0 denote the identity element in A.

For any abelian group A, a graph $G = (V, E)$ is said to be A-magic if there exists a labeling $\ell : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \to A$ defined by

$$\ell^+(v) = \sum\{\ell(uv) : uv \in E(G)\}$$

is a constant map [2]. If $\ell : E(G) \to A \setminus \{0\}(\ |A| > 2)$ is a magic labeling of G with sum c, then $-\ell : E(G) \to A \setminus \{0\}$, defined by $(-\ell)(u) = -\ell(u)$ is another A- magic labeling of G with sum $-c$. The labeling $-\ell$ is called the inverse of ℓ. This implies that A- magic labeling of a graph need not be unique. A graph $G = (V, E)$ is called non-magic if for every abelian group A, the graph is not A-magic [2]. The most obvious example of a non-magic graph is $P_n(n > 3)$, the path of order n. As a result, any graph with a path pendant of length at least two would be non-magic.

The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. Note that V_4 is not cyclic, since every element has order 2 (except for the identity, of course) and V_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The V_4 magic graphs was introduced by S. M. Lee et al. in 2002 [2]. There has been increased interest in the study of V_4 magic graphs since the publication of [2]. We define, a graph G is an a-sum V_4 magic if there exists a labeling $\ell : E(G) \to V_4 \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \to A$ satisfies $\ell^+(v) = a$ for all $v \in V(G)(\ a \neq 0)$. If $\ell^+(v) = 0$, for all $v \in V(G)$, we say that the graph is zero-sum V_4 magic. We divide the class of V_4 magic graphs into the following three categories:

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

Note that $K_3 \in \mathscr{V}_0$, but $K_3 \notin \mathscr{V}_a$ and, $C_4 \in \mathscr{V}_{a,0}$. In this paper, we identify some cycle related graphs which belong to the above three categories.

2 Cycle Related Graphs

Definition 2.1. A path is an ordered list of distinct vertices $u_1, u_2, \ldots, u_n$ such that $u_{i-1}u_i$ is an edge for all i, $2 \leq i \leq n$. The ordered list of vertices $u_1, u_2, \ldots, u_n$ is a cycle if $u_n u_1$ is also an edge. Paths and cycles of n vertices are often denoted by P_n and C_n, respectively. Throughout this paper the suffices of the vertices u_i in the cycle C_n are taken modulo n.

We start with the following lemma.

Lemma 2.1. *If* $\ell : E(C_n) \to V_4 \setminus \{0\}$ *is a labeling of* C_n*, then*

$$\sum_{i=1}^{n} \ell^+(u_i) = 0,$$

where $u_1, u_2, \ldots, u_n$ *are the vertices of* C_n.

Proof. Let ℓ be a labeling of C_n. Then we have, $\ell^+(u_i) = \ell(u_{i-1}u_i) + \ell(u_i u_{i+1})$. This implies that $\sum_{i=1}^{n} \ell^+(u_i) = 0$. This completes the proof. ☐

Using lemma 2.1, we prove the following result:

Theorem 2.1. *$C_n \in \mathscr{V}_a$ if and only if n is even.*

Proof. Assume that $C_n \in \mathscr{V}_a$. Then by lemma 2.1, we have $na = 0$. This implies that n is even. Conversely, assume that n is even. Define $\ell : E(C_n) \to V_4 \setminus \{0\}$ by

$$\ell(u_i u_{i+1}) = b \quad \text{for } i = 1, 3, \ldots, n-1,$$
$$\ell(u_i u_{i+1}) = c \quad \text{for } i = 2, 4, \ldots, n.$$

Obviously, $\ell^+(u_i) = b + c = a$, for $i = 1, 2, \ldots, n$. Thus ℓ is an a-sum V_4 magic labeling of C_n. This completes the proof. □

Theorem 2.2. *$C_n \in \mathscr{V}_0$ for all $n \geq 3$.*

Proof. If we label all the edges of C_n by a, then we obtain $\ell^+(u_i) = 0$ for $i = 1, 2, \ldots, n$. □

Theorem 2.3. *If n is even, then $C_n \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 2.1 and 2.2. □

Definition 2.2. We denote by $C(n, \underbrace{k_1, k_2, \ldots, k_t}_{t})$ the set of all graphs obtained by identifying the apex vertices of t stars K_{1,k_i} $(i = 1, 2, \ldots, t)$ with t $(1 \leq t \leq n)$ vertices of C_n. Observe that $C(n, \underbrace{k, k, \ldots, k}_{n})$ is a unique graph. Four members of $C(16, 7, 7, 7, 7)$ are shown in figures 1and 2. Also two members of $C(15, 7, 7, 7)$ are shown in figure 3.

Theorem 2.4. *If $C(n, k_1, k_2, \ldots, k_t) \subset \mathscr{V}_a$, then $n + k_1 + k_2 + \cdots + k_t$ is even.*

Proof. Observe that each member of $C(n, k_1, k_2, \ldots, k_t)$ has $n + k_1 + k_2 + \cdots + k_t$ vertices. One can verify that $(n + k_1 + k_2 + \cdots + k_t)a = 0$. This implies that $n + k_1 + k_2 + \cdots + k_t$ is even. □

Conjecture 2.5. *If $n + k_1 + k_2 + \cdots + k_t$ is even, then $C(n, k_1, k_2, \ldots, k_t) \subset \mathscr{V}_a$.*

We prove some special cases of conjecture 2.5.

Theorem 2.6. *If $n + tk$ is even, then $C(n, \underbrace{k, k, \ldots, k}_{t}) \subset \mathscr{V}_a$.*

Proof. Consider a graph G in the set $C(n, \underbrace{k, k, \ldots, k}_{t})$. We consider 4 cases:

Case 1: Suppose n, k and t are even. In this case, we label all edges of C_n as described in the proof of theorem 2.1 and label all pendant edges by a. Then obviously this is an a-sum V_4 magic labeling of G.

Case 2: Suppose n, k are even and t is odd. In this case, the labeling is exactly similar to case 1.

Case 3: Suppose n and t are even and k is odd. Without loss of generality assume that apex vertices of the t stars are at $u_1, u_{i_1}, u_{i_2}, \ldots, u_{i_t}$, $1 < i_1 < i_2 < \ldots < i_t$ of the cycle C_n. First, label all pendant edges by a. We label the edges of C_n as follows:
Consider the vertex u_{i_1}. If i_1 is even, we label the edges $u_n u_1, u_1 u_2, \ldots, u_{i_1} u_{i_1+1}$ as follows

$$\begin{aligned} \ell(u_n u_1) &= b, \\ \ell(u_i u_{i+1}) &= b, \text{ for } i = 1, 3, \ldots, i_1 - 1 \\ \ell(u_i u_{i+1}) &= c, \text{ for } i = 2, 4, \ldots, i_1 - 2 \\ \ell(u_{i_1}, u_{i_1+1}) &= b. \end{aligned}$$

If i_1 is odd, we label the edges $u_nu_1, u_1u_2, \ldots, u_{i_1}u_{i_1+1}$ as follows

$$\begin{aligned}\ell(u_nu_1) &= b,\\ \ell(u_iu_{i+1}) &= b, \text{ for } i = 1, 3, \ldots, i_1 - 2\\ \ell(u_iu_{i+1}) &= c, \text{ for } i = 2, 4, \ldots, i_1 - 1\\ \ell(u_{i_1}, u_{i_1+1}) &= c.\end{aligned}$$

So, we have

$$\ell(u_{i_1-1}u_{i_1}) = \ell(u_{i_1}u_{i_1+1}) = \begin{cases} b & \text{if } i_1 \text{ even}\\ c & \text{if } i_1 \text{ odd}\end{cases}$$

Therefore,

$$\ell^+(u_{i_1}) = \begin{cases} b + b + ta = a & \text{if } i_1 \text{ even}\\ c + c + ta = a & \text{if } i_1 \text{ odd,}\end{cases}$$

Furthermore,

$$\ell^+(u_i) = \begin{cases} b + b + ka = a & \text{if } i = 1,\\ b + c = a & \text{if } i = 2, 3, \ldots, i_1 - 1,\\ b + b + ka = a & \text{if } i = i_1 \text{ and } i_1 \text{ is even,}\\ c + c + ka = a & \text{if } i = i_1 \text{ and } i_1 \text{ is odd.}\end{cases}$$

Next, consider the vertex $u_{i_{i_2}}$. Here we consider the following cases:

i_1 and i_2 are even: In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $c, b, c, b, \ldots, c, b, c, c$.

i_1 is even and i_2 is odd : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $c, b, c, b, \ldots, c, b, b$.

i_1 is odd and i_2 is even : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $b, c, b, c, \ldots, b, c, c$.

i_1 and i_2 are odd : In this case we label the edges $u_{i_1+1}u_{i_1+2}, u_{i_1+2}u_{i_1+3}, \ldots, u_{i_2}u_{i_2+1}$ consecutively by $b, c, b, c, \ldots, c, b, b$.

Proceeding like this, we eventually arrive at u_{i_t}. If i_t is even, then obviously $\ell(u_{i_t}u_{i_t-1}) = \ell(u_{i_t}u_{i_t+1}) = c$. Then label the edges $u_{i_t+1}u_{i_t+2}, u_{i_t+2}u_{i_t+3}, \ldots, u_{n-2}u_{n-1}$ consecutively by $b, c, b, c, \ldots, b, c$. If i_t is odd, then obviously $\ell(u_{i_t}u_{i_t-1}) = \ell(u_{i_t}u_{i_t+1}) = b$. Then label the edges $u_{i_t+1}u_{i_t+2}, u_{i_t+2}u_{i_t+3}, \ldots, u_{n-2}u_{n-1}$ consecutively by $c, b, c, b, \ldots, c$. Obviously this labeling is an a-sum v_4 magic labeling of G.

case 4: n, k and t are odd. In this case, the labeling is similar to case 3. Some members of $C(16, 7, 7, 7, 7)$, $C(15, 7, 7, 7)$ and its labelings are shown in figures 1,2 and 3.

This completes the proof. □

Definition 2.3. The corona $G_1 \odot G_2$ of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 [3].

Definition 2.4. The sun on $2n$ vertices is a corona of the form $C_n \odot K_1$ where $n \geq 3$. The sun $C_n \odot K_1$ is denoted by Sun_n. A broken sun is a connected unicyclic subgraph of a sun. We denote by $BS(p, q)$ the set of broken suns with $n = p + q$ vertices and with a p-cycle, note that $BS(p; p) = C_p \odot K_1$. For $p > 2$ and $0 < q < p$, a consecutive broken sun, denoted by $CBSun_{p,q}$ is the graph belonging to $BS(p, q)$ such that the subgraph induced by the vertices of degree 2 is a path on $p - q$ vertices. A broken sun (or a sun) is odd (resp. even) if p is odd (resp.even)[3]. Some examples are depicted in Figure 4.

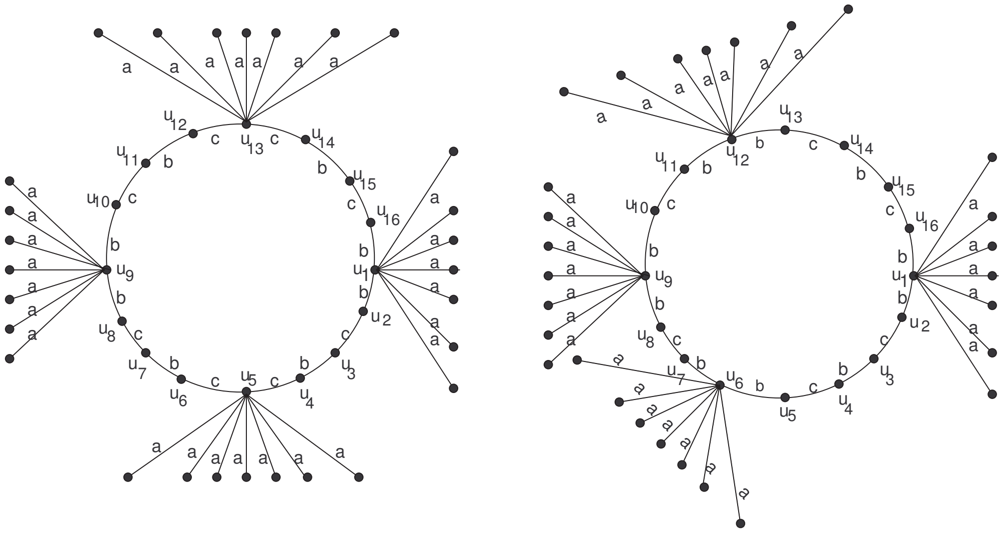

Figure 1: Figure showing members of $C(16, 7, 7, 7, 7)$ and its labelings

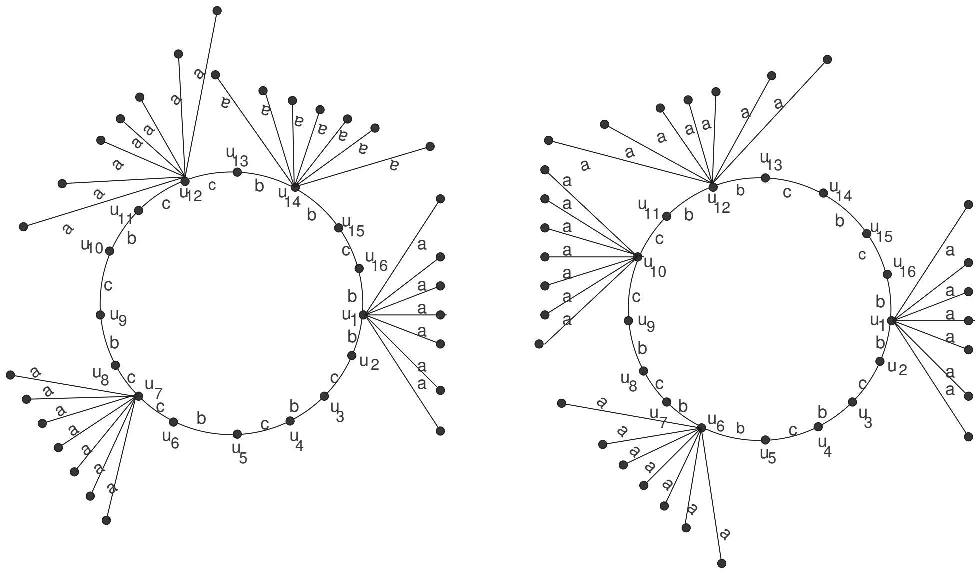

Figure 2: Figure showing members of $C(16, 7, 7, 7, 7)$ and its labelings

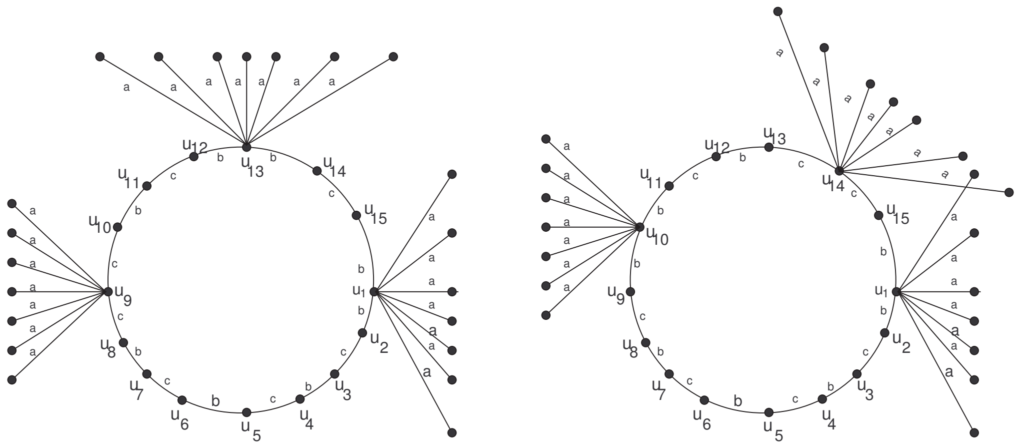

Figure 3: Figure showing some members of $C(15, 7, 7, 7)$ and its labelings

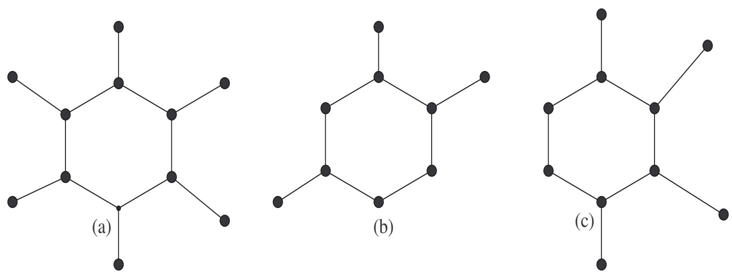

Figure 4: (a)Sun_6, (b)$BSun(6, 3)$, (c) $CBSun(6, 4)$

Theorem 2.7. $Sun_n \in \mathscr{V}_a$ *for all* n.

Proof. Observe that Sun$_n$ $= C(n, \underbrace{1, 1, \ldots, 1}_{n})$. So, the proof of the theorem follows from theorem 2.6. □

Theorem 2.8. $Sun_n \notin \mathscr{V}_0$ *for all* n.

Proof. Since Sun_n has pendant edges, $Sun_n \notin \mathscr{V}_0$. □

Theorem 2.9. $BSun(p, q) \subset \mathscr{V}_a$*if and only if* $p + q$ *is even.*

Proof. Observe that any memeber in $BSun(p, q)$ has $p + q$ number of vertices. If $BSun(p, q) \in \mathscr{V}_a$, then $(p + q)a = 0$. This implies that $p + q$ is even. Converse part follows from theorem 2.6. □

Theorem 2.10. $CBSun(p, q) \subset \mathscr{V}_a$*if and only if* $p + q$ *is even.*

Proof. Proof follows from theorem 2.6. □

Theorem 2.11. *$C_n \odot K_2 \in \mathscr{V}_a$ if and only if n is even.*

Proof. Note that $C_n \odot K_2$ has $3n$ vertices. If $C_n \odot K_2 \in \mathscr{V}_a$, then $3na = 0$. This implies that $na = 0$. Consequently, n is even. Converse part is trivial. □

Theorem 2.12. *$C_n \odot K_2 \in \mathscr{V}_0$ for all $n \geq 3$.*

Proof. Obvious. □

Theorem 2.13. *If n is even, then $C_n \odot K_2 \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 2.11 and 2.12. □

Theorem 2.14. *$C_n \odot C_m \in \mathscr{V}_a$ if and only if $n(m+1)$ is even.*

Proof. Suppose $C_n \odot C_m \in \mathscr{V}_a$. Then, $n(m+1)a = 0$. This implies that $n(m+1)$ is even.

Conversely assume that $n(m+1)$ is even. Let the vertices of C_n be $(u_1, u_2, \ldots, u_n, u_1)$ and let the vertices of C_m be $v_1, v_2, \ldots, v_m$. We consider the following cases

Case 1: Assume that n is even and m is odd. Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$\ell(u_i u_{i+1}) = a$ for $i = 1, 2, \ldots, n,$
$\ell(v_i v_{i+1}) = a$ for $i = 1, 2, \ldots, m.$
for $j = 1, 2, \ldots, n:$
  $\ell(u_j v_i) = a$ for $i = 1, 2, \ldots, m.$
end for

Obviously ℓ is an a-sum V_4 magic labeling of $C_n \odot C_m$.

Case 2: Suppose m and n are odd. In this case the labeling is exactly similar to case 1.

Case 3: Suppose both m and n are even. Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$\ell(u_i u_{i+1}) = b$ for $i = 1, 3, \ldots, n-1,$
$\ell(u_i u_{i+1}) = c$ for $i = 2, 4, \ldots, n,$
$\ell(v_i v_{i+1}) = a$ for $i = 1, 2, \ldots, n.$
for $j = 1, 2, \ldots, n:$
  $\ell(u_j v_i) = a$ for $i = 1, 2, \ldots, m.$
end for

This completes the proof. The graph $C_{12} \odot C_5$ is shown in figure 5. □

Theorem 2.15. *$C_n \odot C_m \in \mathscr{V}_0$ for all $m \geq 3$ and $n \geq 3$.*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$. We consider the following cases:

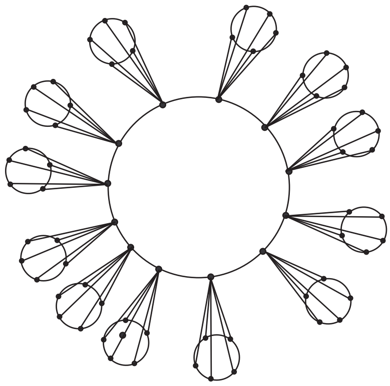

Figure 5: The graph: $C_{12} \odot C_5$

Case 1: Suppose n and m are even. Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$$
\begin{aligned}
&\ell(u_i u_{i+1}) = a \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&\ell(v_i v_{i+1}) = b \quad \text{for} \quad i = 1, 3, \ldots, m-1,\\
&\ell(v_i v_{i+1}) = c \quad \text{for} \quad i = 2, 4, \ldots, m,\\
&\text{for } \; j = 1, 2, \ldots, n:\\
&\qquad \ell(u_j v_i) = a \quad \text{for} \quad i = 1, 2, \ldots, m.\\
&\text{end for}
\end{aligned}
$$

Obviously ℓ is a zero sum V_4 magic labeling of $C_n \odot C_m$.

Case 2: Suppose n is even and m is odd: Define $\ell : E(C_n \odot C_m) \to V_4 \setminus \{0\}$ by

$$
\begin{aligned}
&\ell(u_i u_{i+1}) = a \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&\ell(v_i v_{i+1}) = b \quad \text{for} \quad i = 1, 3, \ldots, m-2,\\
&\ell(v_i v_{i+1}) = c \quad \text{for} \quad i = 2, 4, \ldots, m-1,\\
&\ell(v_m v_1) = a,\\
&\text{for } \; j = 1, 2, \ldots, n:\\
&\qquad \ell(u_j v_i) = a \quad \text{for} \quad i = 1, 2, \ldots, m-1,\\
&\qquad \ell(u_j v_m) = b,\\
&\qquad \ell(u_j v_1) = c.\\
&\text{end for}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\ell(u_i) = a + a + (m-2)a + b + c = 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&\ell(v_i) = a + b + c = 0, \quad \text{for } i = 1, 2, \ldots, m.
\end{aligned}
$$

Case 3: Suppose m and n are odd. In this case the labeling is exactly similar to case 2.

Case 4: Suppose n is odd and m is even. In this case label all edges of $C_n \odot C_m$ by a.

This completes the proof. □

Theorem 2.16. *If $n(m+1)$ is even, then $C_n \odot C_m \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from theorems 2.14 and 2.15. □

Theorem 2.17. *$C_n \odot K_m \in \mathscr{V}_a$ if and only if $n(m+1)$ is even.*

Proof. Observe that $C_n \odot K_m$ has $n + mn$ vertices. If $C_n \odot K_m \in \mathscr{V}_a$, then we have $(m+1)na = 0$. This implies that $n(m+1)$ is even. We consider 3 cases:
Let the vertices of C_n be $u_1, u_2, \ldots, u_n$. We denote the j^{th} copy of K_m by K_m^j. Let the vertices of K_m^j be $\{v_{j,1}, v_{j,2}, \ldots, v_{j,m}\}$.

Case 1: Suppose n is even and m is odd. In this case, first we label all edges of K_m^j by a, $j = 1, 2, \ldots, n$. Next, label all edges of C_n by a. Finally, labell all edges $u_i u_{j,r}$ by a for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $r = 1, 2, \ldots, m$. Obviously, this is an a-sum V_4 magic labeling of $C_n \odot K_m$.

Case 2: Suppose n is even and m is even. In this case, first we label all edges of K_m^j by b, $j = 1, 2, \ldots, n$. Next, label all edges of C_n by $b, c, b, c, \ldots$ consecutively. Finally, labell all edges $u_i u_{j,r}$ by b for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $r = 1, 2, \ldots, m$. Obviously, this is an a-sum V_4 magic labeling of $C_n \odot K_m$.

Case 3: Suppose n and m are odd. In this case, first we label all edges of K_m^j by b, $j = 1, 2, \ldots, n$. Next, label all edges of C_n by a. Finally, label all edges $u_i v_{j,r}$ by a for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $r = 1, 2, \ldots, m$. Obviously, this is an a-sum V_4 magic labeling of $C_n \odot K_m$.

This completes the proof. □

Theorem 2.18. *$C_n \odot \overline{K_m} \in \mathscr{V}_a$ if and only if $n(m+1)$ is even, where $\overline{K_m}$ is the complement of the complete graph with m vertices.*

Proof. Note that the graph $C_n \odot \overline{K_m}$ has $n(m+1)$ vertices. If $C_n \odot \overline{K_m} \in \mathscr{V}_a$, then we have $n(m+1)$ is even. Conversely, assume that $n(m+1)$ is even. Consider n copies of $\overline{K_m}$. Let $\overline{K_m}^j$ denotes the j^{th} copy of $\overline{K_m}$. Let

$$V(C_n) = \{u_1, u_2, \ldots, u_n\},$$
$$V(\overline{K_m}^j) = \{v_{j,1}, v_{j,2}, \ldots, v_{j,m}\},\ j = 1, 2, \ldots, n.$$

We consider 3 cases:

Case 1: Suppose n is even and m is odd. Define $\ell : V(C_n \odot \overline{K_m}) \to V_4 \setminus \{0\}$ by

for $i = 1, 2, \ldots, n$:
$\quad \ell(u_i v_{j,r}) = a,\ j = 1, 2, \ldots, n; r = 1, 2, \ldots, m$
$\quad \ell(u_i u_{i+1}) = a$
end for

Then, we have

$$\ell^+(u_i) = a + a + ma = a$$
$$\ell^+(v_{j,r}) = a,\ j = 1, 2, \ldots, n; r = 1, 2, \ldots, m.$$

Case 2: Suppose n and m are even. Define $\ell : V(C_n \odot \overline{K_m}) \to V_4 \setminus \{0\}$ by

$$\ell(u_iu_{i+1}) = b, \text{ for } i = 1, 3, \ldots, n-1,$$
$$\ell(u_iu_{i+1}) = c, \text{ for } i = 2, 4, \ldots, n$$

for $i = 1, 2, \ldots, n$:

$$\ell(u_iv_{j,r}) = a, \; j = 1, 2, \ldots, n; r = 1, 2, \ldots, m$$

end for

Then, we have

$$\ell^+(u_i) = b + c + ma = a$$
$$\ell^+(v_{j,r}) = a, \; j = 1, 2, \ldots, n; r = 1, 2, \ldots, m.$$

Case 3: Suppose n and m are odd. In this case, the labeling is exactly similar to case 1.

This completes the proof. □

Theorem 2.19. *$C_n \odot \overline{K_m} \notin \mathscr{V}_0$ for all m and n.*

Proof. Obvious. □

Theorem 2.20. *If $n(m+1)$ is even, then $C_n \odot \overline{K_m} \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from 2.18 and 2.19. □

A graph G with a fixed vertex $u \in V(G)$ will be denoted by the ordered pair (G, u). Given two ordered pairs (G, u) and (H, v), one can construct another graph by linking these two graphs through identifying the vertices u and v. We will use the notation $(G, u) \diamond (H, v)$ for this construction or simply $G \diamond H$ if there is no ambiguity regarding the choices of u and v [4].

Definition 2.5. Given n graphs $G_i (i = 1, 2, \ldots, n)$, the chain $G_1 \diamond G_2 \diamond \ldots G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1}. If $G_i = G$, we use the notation $\diamond G_n$ for the n -link chain all of whose links are G[4].

Theorem 2.21. *$C_m \diamond C_n \in \mathscr{V}_a$ if and only if $m + n$ is odd.*

Proof. Let the vertices of C_m and C_n be respectively, $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$. Assume that u_1 and v_1 are identified with a new vertex w. Then we have, $\sum_{i=2}^{m} \ell^+(u_i) + \sum_{i=2}^{n} \ell^+(v_i) + \ell^+(w) = 0$. This implies that $(m + n)$ is odd.

Conversely, assume that $m + n$ is odd. Then we consider two cases:

Case 1: Suppose m is even and n is odd. Define a mapping $\ell : E(C_m \diamond C_n) \to V_4 \setminus \{0\}$ by

$$\ell(u_iu_{i+1}) = \begin{cases} b & \text{for} \quad i = 1, 3, \ldots, m-1, \\ c & \text{for} \quad i = 2, 4, \ldots, m, \end{cases}$$

$$\ell(v_iv_{i+1}) = \begin{cases} c & \text{for} \quad i = 1, 3, \ldots, n, \\ b & \text{for} \quad i = 2, 4, \ldots, n-1. \end{cases}$$

Clearly ℓ is an a-sum magic labeling of $C_m \diamond C_n$.

Case 2: Suppose m is odd and n is even. The remaining part is exactly similar to case 1.

This completes the proof. □

Theorem 2.22. *$\diamond[C_n]_m \in \mathscr{V}_a$ if and only if m is odd and n is even.*

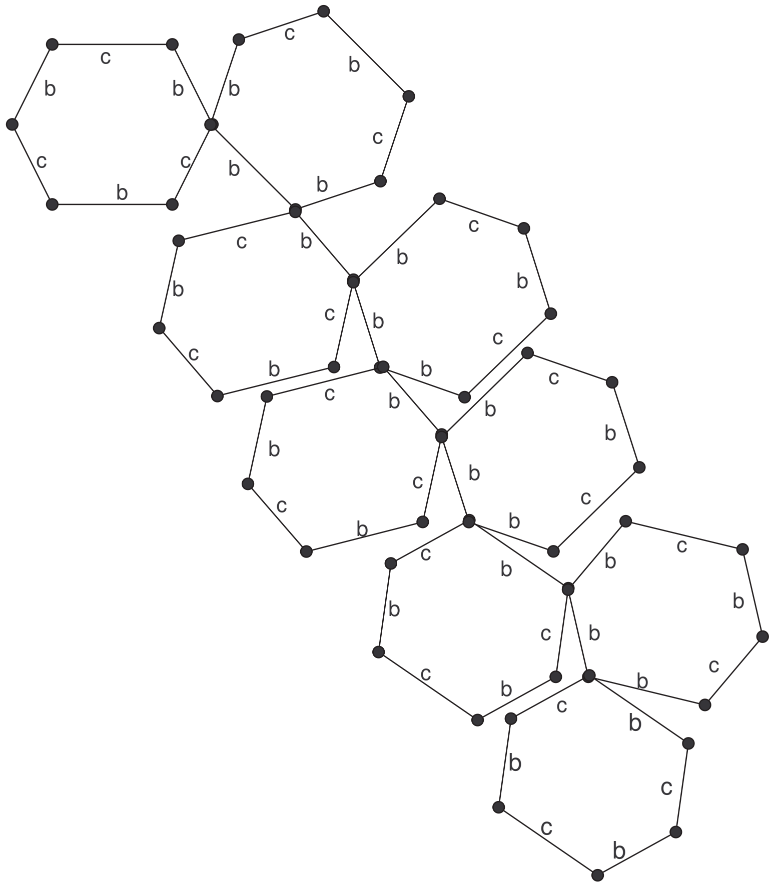

Figure 6: An a-sum V_4 magic labeling of $\diamond[C_6]_9$

Proof. Observe that $\diamond[C_n]_m$ has $mn-m+1$ vertices. If $\diamond[C_n]_m \in \mathscr{V}_a$, then we have $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. Consequently, m is odd and n even.

Conversely, assume that n is even ans m is odd. Consider m copies of C_n. Let the vertices of the i^{th} cycle C_n^i be $(u_1^i, u_2^i, \ldots, u_n^i, u_1^i)$. First, consider the pairs (C_n^1, u_1^1) and (C_n^2, u_1^2) and construct $G_1 = (C_n^1, u_1^1) \diamond (C_n^2, u_1^2)$. Next consider the pairs (G_1, u_2^2) and (C_n^3, u_1^3) and construct $G_2 = (G_1, u_2^2) \diamond (C_n^3, u_1^3)$. Proceeding like this, we finally arrive at $G_{m-1} = (G_{m-2}, u_2^{m-1}) \diamond (C_n^m, u_1^m)$. We need to show that $G = G_1 \diamond G_2 \diamond \cdots \diamond G_{m-1} \in \mathscr{V}_a$. We label the edges of G by the following table:

$i\backslash$edge	$u_1^iu_2^i$	$u_2^iu_3^i$	$u_3^iu_4^i$	$u_4^iu_5^i$	$u_5^iu_6^i$	$u_6^iu_7^i$	$\ldots$	$u_{n-1}^iu_n^i$	$u_n^iu_1^i$
1	b	c	b	c	b	c	$\ldots$	b	c
2	b	b	c	b	c	b	$\ldots$	c	b
3	b	c	b	c	b	c	$\ldots$	b	c
4	b	b	c	b	c	b	$\ldots$	c	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
m	b	c	b	c	b	c	$\ldots$	b	c

One can easily verify that this is a a-sum V_4 magic labeling of G. This completes the proof. An a-sum V_4 magic labeling of $\diamond[C_6]_9$ is shown in figure 6. ☐

Theorem 2.23. *$C_m \diamond C_n \in \mathscr{V}_0$ for all m and n.*

Proof. Label all edges by a, we obtain $\ell^+ \equiv 0$. ☐

Theorem 2.24. *If $m+n$ is odd, then $C_m \diamond C_n \in \mathscr{V}_{a,0}$.*

Proof. Proof follows from 2.21 and 2.23. ☐

Theorem 2.25. *$\diamond C_n \in \mathscr{V}_0$.*

Proof. If we label all edges of $\diamond C_n$ by a, we obtain a zero sum V_4 magic labeling of $\diamond C_n$. ☐

Definition 2.6. A wheel graph denoted by W_n is defined as $W_n \simeq C_n + K_1$, where C_n for $n \geq 3$ is a cycle of length n [4].

Here we need the following lemma.

Lemma 2.2. *If $\ell : E(W_1) \to V_4 \setminus \{0\}$ is a labeling of W_n, then*

$$\sum_{i=1}^{n} \ell^+(u_i) = \ell^+(u) \tag{2.1}$$

where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle C_n and u is the central vertex of W_n.

Proof. Observe that

$$\ell^+(u) = \sum_{i=1}^{n} \ell(uu_i), \tag{2.2}$$

and

$$\ell^+(u_i) = \ell(u_{i-1}u_i) + \ell(u_iu_{i+1}) + \ell(uu_i). \tag{2.3}$$

Therefore

$$\sum_{i=1}^{n} \ell^+(u_i) = 2\sum_{i=1}^{n} \ell(u_iu_{i+1}) + \sum_{i=1}^{n} \ell(uu_i)$$

That is,

$$\sum_{i=1}^{n} \ell^{+}(u_i) - \ell^{+}(u) = 2\sum_{i=1}^{n} \ell(u_i u_{i+1}). \tag{2.4}$$

Note that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) \in V_4$. Therefore, $2\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. Hence equation (2.4) reduces to

$$\sum_{i=1}^{n} \ell^{+}(u_i) = \ell^{+}(u).$$

This completes the proof. ☐

Theorem 2.26. *$W_n \in \mathscr{V}_a$ if and only if n is odd.*

Proof. Suppose W_n admits an a- sum V_4 magic labeling. Then by lemma, we have

$$na = a, \ a \neq 0.$$

This implies that n is odd.

Conversely, assume that n is odd. We will prove that W_n admits an a-sum V_4 magic labeling. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be a labeling of W_n such that $\ell(uu_i) = a$ for all i . Since n is odd, $\sum_{i=1}^{n} \ell(uu_i) = a$. Thus $\ell^{+}(u) = a$. Note that $\ell(u_i u_{i+1}) \in V_4$ for $i = 1, 2, \ldots, n$, where $u_{n+1} = u_1$. Therefore, $2\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. This implies that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0, a, b$ or c. Without loss of generality assume that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. The other cases are similar. Note that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$ can be written as:

$$\ell(u_1 u_2) + \sum_{i=2}^{n} \ell(u_i u_{i+1}) = 0. \tag{2.5}$$

Let us take $\ell(u_1 u_2) = a$. One can assign b or c to $\ell(u_1 u_2)$ instead of a. If $\ell(u_1 u_2) = a$, the second term in equation (2.5) can be taken as a. That is,

$$\sum_{i=2}^{n} \ell(u_i u_{i+1}) = a. \tag{2.6}$$

Note that equation (2.6) can be written as:

$$\ell(u_2 u_3) + \sum_{i=3}^{n} \ell(u_i u_{i+1}) = a. \tag{2.7}$$

For an a-sum V_4 magic graph, we need

$$\ell(u_1 u_2) + \ell(u_2 u_3) + \ell(uu_2) = a.$$

This equation implies that $\ell(u_2 u_3) = a$. Hence $\ell^{+}(u_2) = a$. From equation (2.7), we have $\sum_{i=3}^{n} \ell(u_i u_{i+1}) = 0$. That is,

$$\sum_{i=3}^{n} \ell(u_i u_{i+1}) = 0 \tag{2.8}$$

Again, equation (2.8) can be written as:

$$\ell(u_3 u_4) + \sum_{i=4}^{n} \ell(u_i u_{i+1}) = 0. \tag{2.9}$$

For an a-sum V_4 magic graph, we need

$$\ell(u_3 u_4) + \ell(uu_3) + \ell(u_2 u_3) = a$$

This implies that $\ell(u_3 u_4) = a$. Hence $\ell^{+}(u_3) = a$, If we continue this process we finally arrive at $\ell(u_n u_1) = a$ and $\ell^{+}(u_1) = a$. Thus ℓ is an a-sum V_4 magic labeling of W_n. ☐

A step by step procedure for finding an a-sum magic map for W_n, when n is odd is given below:

1. For $i = 1, 2, \ldots$, setting
$$\ell(uu_i) = a \text{ or } b \text{ or } c.$$
2. Consider the equation $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. Assume that $\ell(uu_i) = a$.
3. Split 0 into two parts. We have the following possibilities:
$$a + a = 0, \quad b + b = 0, \quad c + c = 0$$
4. Consider the first sum $a + a = 0$ and take $\ell(u_1 u_2)$ as a. Then $\sum_{i=2}^{n} \ell(u_i u_{i+1}) = a$. One can consider the other two cases also.
5. Split the summation $\sum_{i=2}^{n} \ell(u_i u_{i+1}) = a$ in the following form
$$\ell(u_2 u_3) + \sum_{i=3}^{n} \ell(u_i u_{i+1}) = a.$$
Find the value to $\ell(u_2 u_3)$ from the following equation:
$$\ell(u_1 u_2) + \ell(uu_2) + \ell(u_2 u_3) = a.$$
6. Continue this processes up to the $(n-1)^{\text{th}}$ step. Finally the value of $\ell(u_n u_1)$ is determined by the equation:
$$\ell(u_{n-1} u_n) + \ell(uu_n) + \ell(u_1 u_n) = a$$

Observe that a-sum V_4 magic labeling of W_n is not unique. The following is another procedure for obtaining an a-sum V_4 magic labeling of W_n.

1. Consider the equation
$$\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0, a, b \text{ or } c. \tag{2.10}$$
Without loss of generality assume that
$$\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0. \tag{2.11}$$
The equation (2.11) can be written as:
$$\ell(u_1 u_2) + \sum_{i=2}^{n} \ell(u_i u_{i+1}) = 0. \tag{2.12}$$
Assign a or b or c to $\ell(u_1 u_2)$. Let us assign a to $\ell(u_1 u_2)$. Then from equation (2.12) one obtain,
$$\sum_{i=2}^{n} \ell(u_i u_{i+1}) = a. \tag{2.13}$$
Equation (2.13) can be written as:
$$\ell(u_2 u_3) + \sum_{i=3}^{n} \ell(u_i u_{i+1}) = a. \tag{2.14}$$
Assign any value to $\ell(u_2 u_3)$ from the set $\{a, b, c\}$. Let us assume that $\ell(u_2 u_3) = b$. Choose $\ell(uu_2)$ such that
$$\ell(u_1 u_2) + \ell(uu_2) + \ell(u_2 u_3) = a. \tag{2.15}$$
Hence we have
$$\ell^{+}(u_2) = a.$$

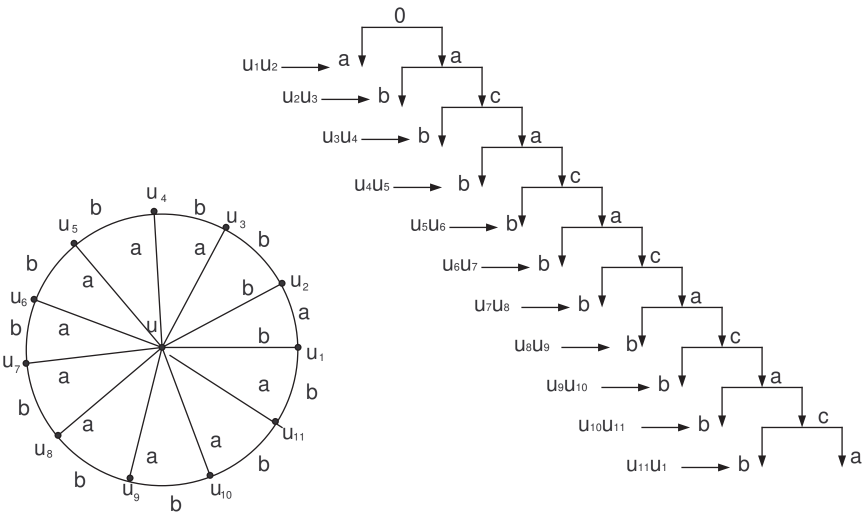

Figure 7: An a-sum V_4 magic labeling of W_{11}

2. From equation (2.13), we have
$$\sum_{i=3}^{n} \ell(u_i u_{i+1}) = c \tag{2.16}$$
Applying the same procedure as explained above, one obtain:
$$\ell^+(u_3) = a.$$
3. Continue the above processes. Finally, we obtain
$$\ell^+(u_1) = a.$$
4. Since ℓ is a labeling of W_n, by lemma 2.2, we have
$$\ell^+(u) = \sum_{i=1}^{n} \ell^+(u_i) = na$$
Since n is odd, we have $na = a$. Therefore, we have $\ell^+(u) = a$.

Several a-sum magic labelings of W_{11} are shown in figure 7 and 8.

Theorem 2.27. *$W_n \in \mathscr{V}_0$, if n is odd.*

Proof. Suppose n is odd. Define $\ell : E(W_n) \to V_4 \setminus \{0\}$ by
$$\ell(uu_i) = a \text{ for } i = 1, 2, \ldots, n-2,$$
$$\ell(uu_i) = c \text{ for } i = n-1,$$
$$\ell(uu_i) = b \text{ for } i = n,$$
$$\ell(u_i u_{i+1}) = b \text{ for } i = 1, 3, \ldots, n-2$$

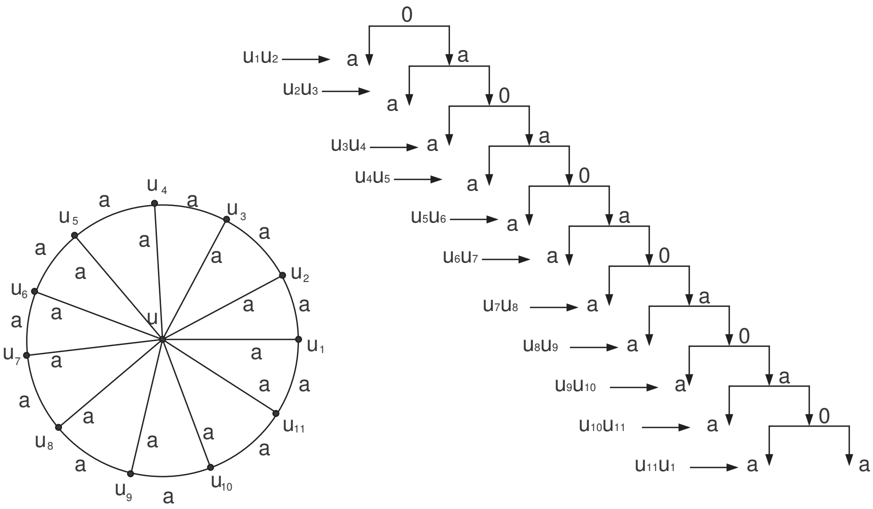

Figure 8: An a-sum V_4 magic labeling of W_{11}

$$\ell(u_i u_{i+1}) = c \text{ for } i = 2, 4, \ldots, n-3$$
$$\ell(u_{n-1}u_n) = a$$
$$\ell(u_n u_1) = c.$$

Obviously ℓ is a zero sum V_4 magic labeling of W_n. □

A zero sum magic labeling of W_7 and W_{13} are shown in figure 9. Further, two different zero sum V_4 magic labeling of W_{11} is shown in figure 10 and figure 11.

Theorem 2.28. *$W_n \in \mathscr{V}_0$, if n is even*

Proof. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be a labeling of W_n. Then $2\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0$. This implies that $\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0, a, b$ or c. Without loss of generality assume that

$$\sum_{i=1}^{n} \ell(u_i u_{i+1}) = 0. \tag{2.17}$$

Rest of the proof is exactly similar to the algorithm for finding the a-sum V_4 magic labeling of W_n explained above subject to the condition that no element will repeat consecutively on the outer circle of W_n. □

Theorem 2.29. *If $n \equiv 0(\text{mod } 3)$, then W_n admits a zero sum V_4 magic labeling.*

Proof. Define $\ell : E(W_n) \to V_4 \setminus \{0\}$ as follows:

$$\ell(u_i u_{i+1}) = b, \ell(uu_i) = c \quad \text{for } i = 1, 4, 7, \ldots, n-2,$$
$$\ell(u_i u_{j+1}) = c, \ell(uu_i) = a \quad \text{for } j = 2, 5, 8, \ldots, n-1,$$
$$\ell(u_i u_{i+1(\text{mod } n)}) = a, \ell(uu_i) = b \quad \text{for } i = 3, 6, 9, \ldots, n.$$

Obviously ℓ is a zero sum magic labeling of W_n. □

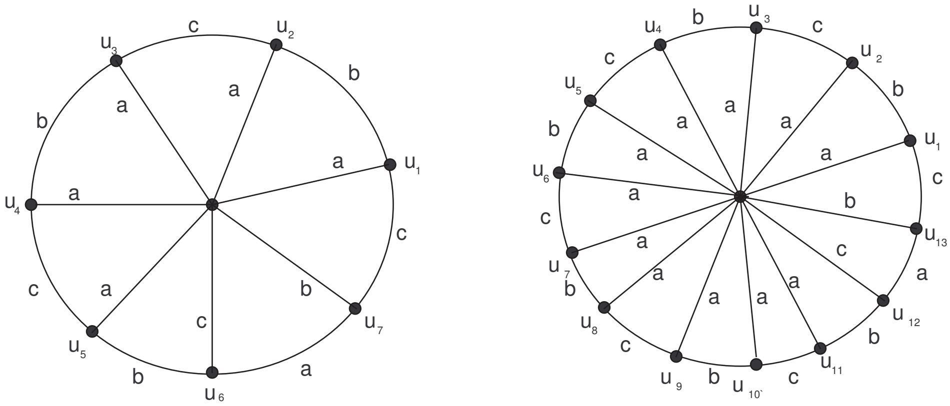

Figure 9: A Zero-sum V_4 magic labeling of W_7 and W_{13}

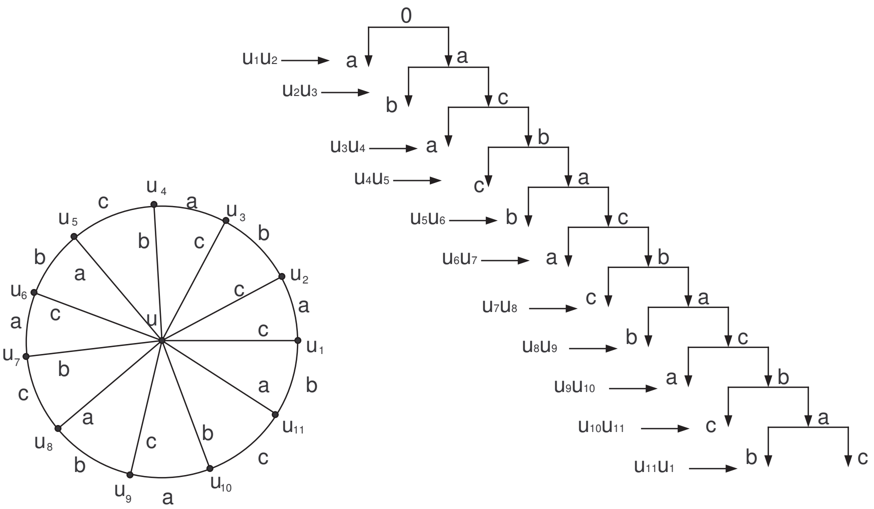

Figure 10: A zero-sum V_4 magic labeling of W_{11}

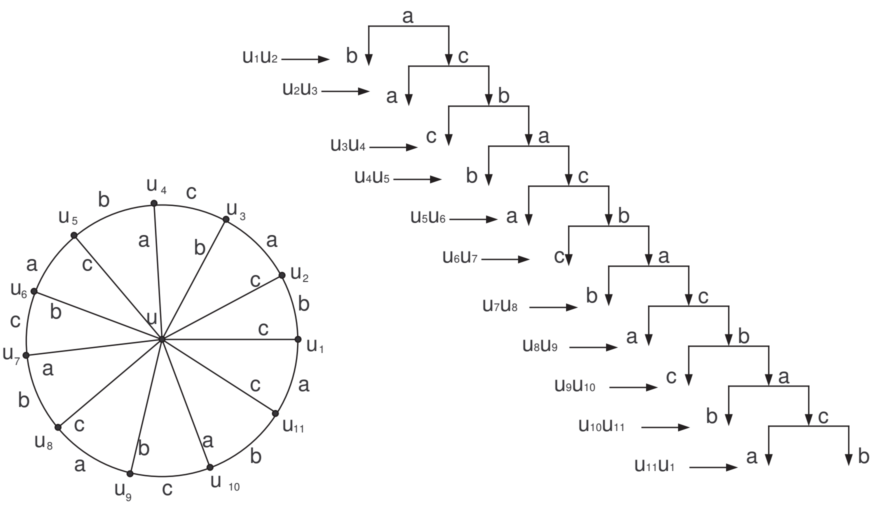

Figure 11: A 0-sum V_4 magic labeling of W_{11}

Theorem 2.30. *If W_n is a- sum V_4 magic and if k is odd, then W_{nk} is a-sum V_4 magic.*

Proof. Assume that W_n is a- sum V_4 magic. Then by theorem 2.26, we have n is odd. Since k is odd this implies that nk is odd. Hence theorem 2.26 tells us that W_{nk} is a-sum V_4 magic. □

Next, we will explain a procedure for obtaining an a-sum V_4 magic labeling W_{nk} if an a-sum V_4 magic labeling of W_n is known.
Let $C_{n,1} : v_1, v_2, v_3, \ldots, v_n, v_1$ and v be the center vertex of W_n. Let $C_{nk,1} : u_1, u_2, \ldots, u_{kn}, u_1$ and u be the center vertex of $W_{nk,1}$. Let $\ell : E(W_n) \to V_4 \setminus \{0\}$ be an a-sum V_4 magic labeling of W_n. Whenever $m \equiv i(\text{mod } n)$, define a function $\ell' : E(W_{nk}) \to V_4 \setminus \{0\}$ by

$$\ell'(uu_m) = \ell(vv_i), \text{ for } m \equiv i(\text{mod } n)$$
$$\ell'(u_m u_{m+1}) = \ell(v_i v_{i+1}), \text{ for } m \equiv i(\text{mod } n).$$

If ℓ'^{+} is the induced vertex labeling of W_{nk}, then $\ell'^{+}(u) = k\ell^{+}(v) = ka = a$ and

$$\ell'^{+}(u_i) = \ell'(u_{m-1}u_m) + \ell'(uu_m) + \ell'(u_m u_{m+1})$$
$$= \ell(u_{(m-1) \text{ mod } n} u_{m \text{ (mod } n)}) + \ell(uu_{m(\text{mod } n)}) + \ell(u_{m(\text{mod } n)} u_{(m+1)(\text{mod } n)}) = a.$$

Hence ℓ' is an a-sum V_4 magic labeling of W_{nk}. An a-sum V_4 magic labeling of W_3 and W_{15} is shown in figure12.

Theorem 2.31. *If W_n is zero-sum V_4 magic, so is W_{kn} for every $k \geq 2$.*

Definition 2.7. A double-wheel graph $W_{n,2}$ can be obtained as join of $2C_n + K_1$, and inductively we can construct an m-level wheel graph denoted by $W_{n,m}$ as follows $W_{n,m} \simeq mC_n + K_1$ [4].

Let $C_{n,1}, \ldots, C_{n,m}$ represent the cycles of $W_{n,m}$ at levels $1, \ldots, m$, respectively, as shown in figure 13.

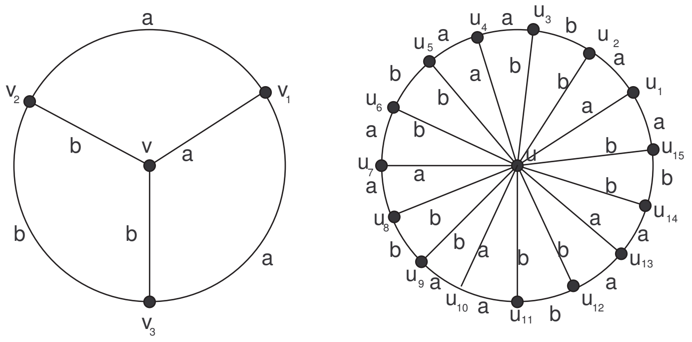

Figure 12: An a-sum V_4 magic labeling of W_3 and W_{15}

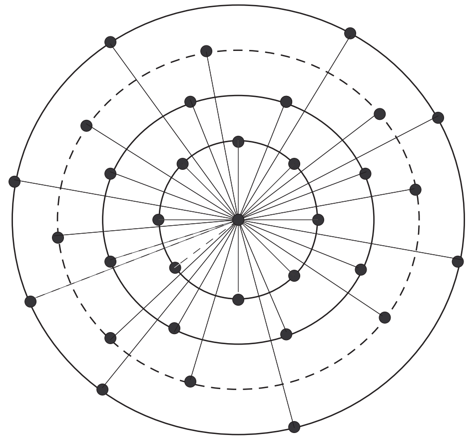

Figure 13: An m- level Wheel: $W_{n,m}$

Lemma 2.3. *If $\ell : E(W_{n,m}) \to V_4 \setminus \{0\}$ is a labeling of $W_{n,m}$, then*

$$\sum_{j=1}^{m}\sum_{i=1}^{n} \ell^+(u_{i,j}) = \ell^+(u) \tag{2.18}$$

where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$ and u is the central vertex of $W_{n,m}$.

Theorem 2.32. *$W_{n,m} \in \mathscr{V}_a$ if and only if both m and n are odd.*

Proof. If $W_{n,m} \in \mathscr{V}_a$, then by lemma 2.3, we have

$$(mn)a = a.$$

This implies that mn is odd or equivalently m and n are both odd.

Conversely, assume that both m and n are odd. If we label all the edges of $W_{n,m}$ by a, then obviously $\ell^+(u) = a$ and $\ell^+(u_{ij}) = a$. □

Theorem 2.33. *$W_{n,m} \in \mathscr{V}_0$ for all m and n.*

Proof. Obvious. □

Theorem 2.34. *If $W_{n,m} \in \mathscr{V}_a$, then $W_{kn,m} \in \mathscr{V}_a$ if k is odd.*

Proof. $W_{n,m} \in \mathscr{V}_a$ implies that mn is odd. This implies that both m and n are odd. Now, $W_{kn,m} \in \mathscr{V}_a$ if mnk is odd. This implies that k is odd. □

Theorem 2.35. *If $W_{n,m} \in \mathscr{V}_0$, then $W_{nk,m} \in \mathscr{V}_0$ for any $k \geq 1$.*

Definition 2.8. A subdivided wheel graph denoted by SW_n is obtained by dividing each spoke uu_i. Similarly, we can define the subdivided m-level graph $SW_{n,m}$.

Lemma 2.4. *If $\ell : E(SW_n) \to V_4 \setminus \{0\}$ is a labeling of SW_n, then*

$$\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(u). \tag{2.19}$$

where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the vertices corresponding the subdivision of the spokes uu_i and u is the central vertex of W_n.

Theorem 2.36. *$SW_n \notin \mathscr{V}_a$ for any $n \geq 3$.*

Proof. Assume that SW_n admits an a-sum V_4 magic labeling. Then by lemma 2.4, we have

$$na + na = a$$

This implies that $a = 0$. □

Theorem 2.37. *$SW_n \in \mathscr{V}_0$ for any $n \geq 3$.*

Proof. We consider two cases:

Case 1: If n is even, define $\ell : E(SW_n) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} \ell(uv_i) &= a, \text{ for } i = 1, 2, 3, \ldots, n, \\ \ell(v_i u_i) &= a, \text{ for } i = 1, 2, 3, \ldots, n, \\ \ell(u_i u_{(i+1)}) &= b, \text{ for } i = 1, 3, \ldots, n-1, \\ \ell(u_i u_{(i+1) \pmod n}) &= c, \text{ for } i = 2, 4, \ldots, n. \end{aligned}$$

Obviously $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathscr{V}_0$ if n is even.

Case 2: If n is odd, define $\ell : E(SW_n) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_i) &= a, \text{ for } i = 1, 2, 3, \ldots, n-2, \\
\ell(uv_i) &= b, \text{ for } i = n-1, \\
\ell(uv_i) &= c, \text{ for } i = n, \\
\ell(v_i u_i) &= a, \text{ for } i = 1, 2, \ldots, n-2, \\
\ell(v_i u_i) &= b, \text{ for } i = n-1, \\
\ell(v_i u_i) &= c, \text{ for } i = n, \\
\ell(u_i u_{(i+1)}) &= b, \text{ for } i = 2, 4, \ldots, n-3, \\
\ell(u_i u_{(i+1)}) &= c, \text{ for } i = 1, 3, \ldots, n-2, \\
\ell(u_{n-1} u_n) &= a, \\
\ell(u_n u_1) &= b.
\end{aligned}$$

Observe that $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathscr{V}_0$ if n is odd.

□

Lemma 2.5. *If $\ell : E(SW_{n,m}) \to V_4 \setminus \{0\}$ is a labeling of $SW_{n,m}$, then*

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \ell^+(u_{i,j}) + \sum_{j=1}^{m} \sum_{i=1}^{n} \ell^+(v_{i,j}) = \ell^+(u). \tag{2.20}$$

where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$, $v_{1,j}, v_{2,j}, \ldots, v_{n,j}$ are the subdivisions corresponding to the edges $uu_{i,j}$ and u is the central vertex.

Theorem 2.38. *$SW_{n,m} \notin \mathscr{V}_a$ for any n and m.*

Proof. Obvious. □

Theorem 2.39. *$SW_{n,m} \in \mathscr{V}_0$ for any n and m.*

Proof. We consider two cases:

Case 1: If n is even, for $j = 1, 2, \ldots, m$, define $\ell : E(SW_{n,m}) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_{i,j}) &= a \text{ for } i = 1, 2, 3, \ldots, n, \\
\ell(v_{i,j} u_{i,j}) &= a \text{ for } i = 1, 2, 3, \ldots, n, \\
\ell(u_{i,j} u_{i+1,j}) &= b, \text{for } i = 1, 3, \ldots, n-1, \\
\ell(u_{i,j} u_{i+1,j}) &= c, \text{for } i = 2, 4, \ldots, n.
\end{aligned}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

Case 2: If n is odd, for $j = 1, 2, \ldots, m$, define $\ell : E(SW_{n,m}) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_{i,j}) &= a, \text{ for } i = 1, 2, 3, \ldots, n-2, \\
\ell(uv_{i,j}) &= b, \text{ for } i = n-1 \\
\ell(uv_{i,j}) &= c, \text{ for } i = n \\
\ell(v_{i,j} u_{i,j}) &= a, \text{ for } i = 1, 2, 3, \ldots, n-2, \\
\ell(u_{i,j} v_{i,j}) &= b, \text{for } i = n-1, \\
\ell(u_{i,j} v_{i,j}) &= c, \text{for } i = n,
\end{aligned}$$

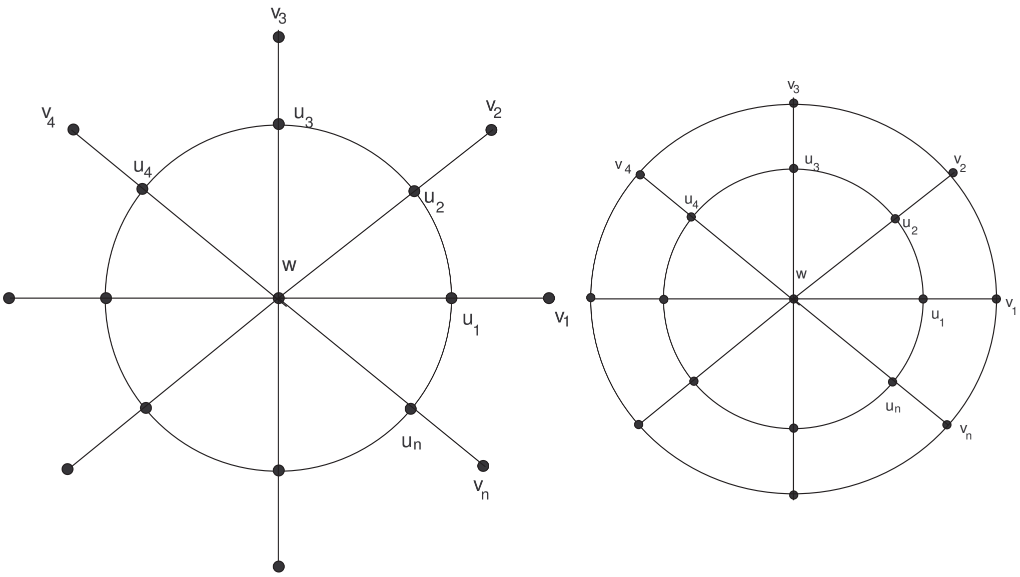

Figure 14: The helm: H_n (left) and the closed helm :$H(2,n)$(right)

$$\begin{aligned}
\ell(u_{i,j}u_{i+1,j}) &= c, \text{for } i = 1, 3, \ldots, n-2\\
\ell(u_{i,j}u_{i+1,j}) &= b, \text{for } i = 2, 4, \ldots, n-3\\
\ell(u_{i,j}u_{n,j}) &= a, \text{for } i = n-1\\
\ell(u_{i,j}u_{1,j}) &= b, \text{for } i = n.
\end{aligned}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

□

Definition 2.9. The helm H_n is the graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the cycle C_n (see figure 14)[5].

Lemma 2.6. *If $\ell : E(H_n) \to V_4 \setminus \{0\}$ is a labeling of H_n, then*

$$\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(u). \tag{2.21}$$

where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the pendant vertices corresponding to the spokes uu_i and u is the central vertex of W_n.

Theorem 2.40. *$H_n \notin \mathscr{V}_a$ for any n.*

Proof. Proof follows from lemma 2.6. □

Theorem 2.41. *$H_n \notin \mathscr{V}_0$ for any n.*

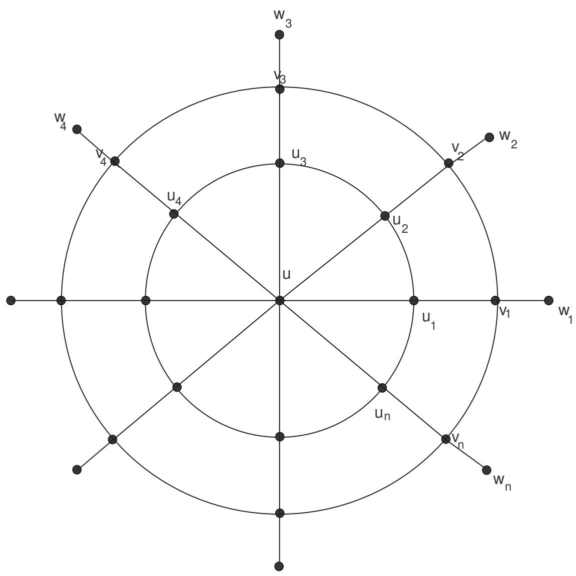

Figure 15: Web graph : $W(2, n)$

Proof. Obvious. □

Definition 2.10. The web graph $W(2, n)$ is the graph obtained by joining the pendant points of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle (see figure 15)[5].

Lemma 2.7. *If $\ell : E(W(2, n)) \to V_4 \setminus \{0\}$ is a labeling of $W(2, n)$, then*

$$\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) + \sum_{i=1}^{n} \ell^+(w_i) = \ell^+(u). \tag{2.22}$$

where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the vertices of $C_{n,2}$, $w_1, w_2, \ldots, w_n$ are the pendant vertices and u is the hub of $W(2, n)$.

Theorem 2.42. *$W(2, n) \in \mathscr{V}_a$ if and only if n is odd.*

Proof. Assume that $W(2, n) \in \mathscr{V}_a$. Then from lemma 2.7, we have

$$na + na + na = a$$

This implies that $na = a$. This equation holds if and only if n is odd.

Conversely, assume that n is odd. Define a mapping $\ell : E(W(2, n)) \to V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(uu_i) &= a \ \text{ for } \ i = 1, 2, \ldots, n, \\ \ell(u_iu_{i+1}) &= b \ \text{ for } \ i = 1, 3, \ldots, n, \\ \ell(u_nu_1) &= a \end{aligned}$$

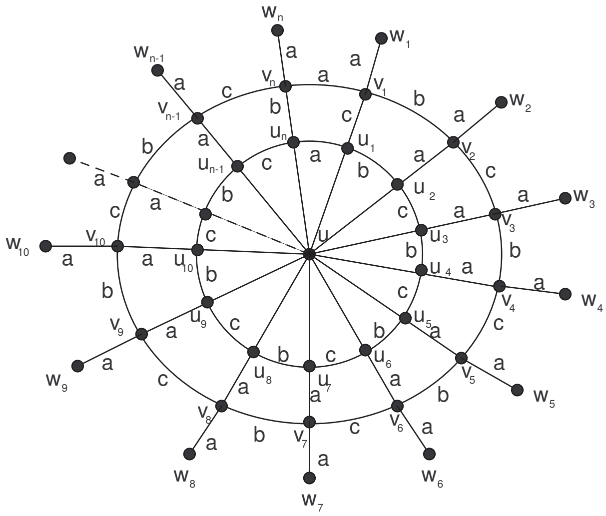

Figure 16: An a-sum V_4 magic labeling of $W(2,n)$

$$
\begin{aligned}
\ell((u_iu_{i+1}) &= c \text{ for } i = 2,4,\ldots,n-1,\\
\ell(v_iv_{i+1}) &= b \text{ for } i = 1,3,\ldots,n,\\
\ell(v_nv_1) &= a\\
\ell((v_iv_{i+1}) &= c \text{ for } i = 2,4,\ldots,n-1,\\
\ell(u_1v_1) &= c,\\
\ell(u_iv_i) &= a \text{ for } i = 2,4,\ldots,n-1,\\
\ell(u_nv_n) &= b,\\
\ell(v_iw_i) &= a.
\end{aligned}
$$

Obviously, $\ell^+(u) = a$ and $\ell^+(u_i) = \ell^+(v_i) = \ell(w_i) = a$ for $i = 1,2,\ldots,n.$ A a-sum V_4 magic labeling is shown in figure 16.

□

Theorem 2.43. $W(2,n) \notin \mathscr{V}_0$ *for any* n.

Proof. Obvious. □

Definition 2.11. The generalized web graph $W(t,n)$ is the graph obtained by iterating the processes of constructing web graph $W(2,n)$ from the helm H_n, so that the web has t n-cycles (See figure 17)[5].

Lemma 2.8. *If* $\ell : E(W(t,n)) \to V_4 \setminus \{0\}$ *is a labeling of* $W(t,n)$*, then*

$$\sum_{i=1}^{t}\sum_{i=1}^{n} \ell^+(u_{i,j}) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(u). \tag{2.23}$$

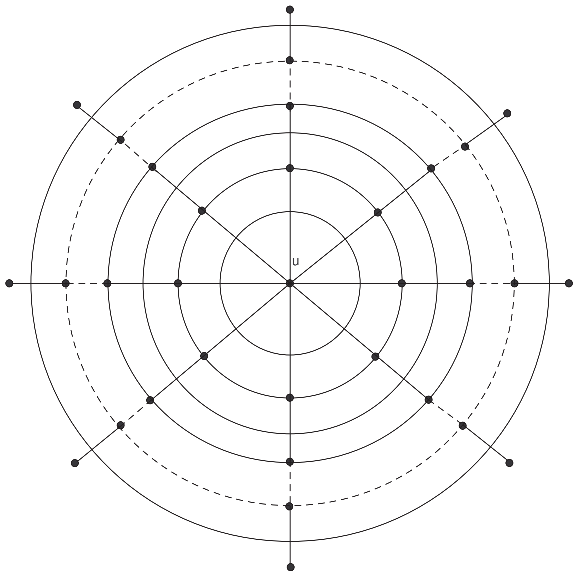

Figure 17: Generalised Web graph : $W(t, n)$,

where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, $j = 1, 2, \ldots, t$ and $v_1, v_2, \ldots, v_n$ are the pendant vertices and u is the hub of $W(2, n)$.

Proof. Obvious. □

Theorem 2.44. *$W(t, n) \in \mathscr{V}_a$ if and only if n is odd and t is even.*

Proof. Assume that $W(t, n) \in \mathscr{V}_a$. Then by lemma 2.8, we have

$$n(t + 1)a = a$$

This implies that n is odd and t is even. Conversely, if n is odd and t is even one can easily prove that $W(t, n) \in \mathscr{V}_a$. □

Theorem 2.45. *$W(t, n) \notin \mathscr{V}_0$ for any n and any t.*

Proof. Obvious. □

Definition 2.12. The generalized web graph without center, $W_0(t, n)$ is the graph obtained by removing the central vertex of $W(t, n)$ [5]. The graph of $W_0(t, n)$ is shown in figure 18.

Lemma 2.9. *If $\ell : E(W_0(t, n)) \to V_4 \setminus \{0\}$ is a labeling of $W_0(t, n)$, then*

$$\sum_{j=1}^{t} \sum_{i=1}^{n} \ell^+(u_{i,j}) + \sum_{i=1}^{n} \ell^+(v_i) = 0. \tag{2.24}$$

where $u_{1,j}, u_{2,j}, \ldots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, and $v_1, v_2, \ldots, v_n$ are the pendant vertices.

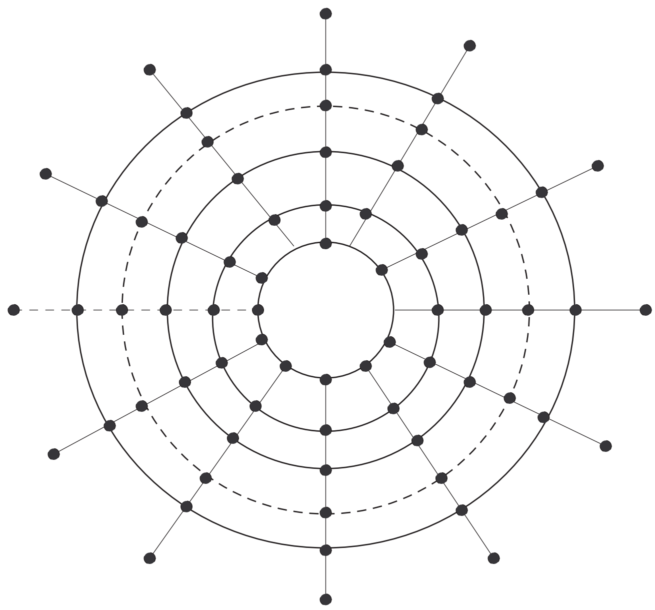

Figure 18: Generalised Web graph without centre: $W_0(t, n)$

Proof. Obvious. □

Theorem 2.46. *If $W_0(t, n) \in \mathscr{V}_a$ if and only if $n(t + 1)$ is even.*

Proof. First, assume that $W_0(t, n) \in \mathscr{V}_a$. Then by lemma 2.9, we have $nta + na = 0$. This implies that $n(t + 1)$ is even.

Conversely, assume that $n(t + 1)$ is even. we consider the following cases:

Case 1: If n and t are even, define $\ell : E(W_0(t, n)) \to V_4 \setminus \{0\}$ as follows:

$\ell(u_{i,1}u_{(i+1)(\text{mod } n),1}) = a$ for $i = 1, 2, 3, \ldots, n$
for $j = 2, 3, \ldots, t$:
  $\ell(u_{i,j}u_{i+1,j}) = c$ for $i = 1, 3, \ldots, n - 1$
  $\ell(u_{i,j}u_{(i+1)(\text{mod } n),j}) = b$ for $i = 2, 4, \ldots, n$
end for
for $j = 1, 2, \ldots, n$:
  $\ell(u_{i,j}u_{i,j+1}) = a$ for $i = 1, 2, \ldots, t - 1$
end for
$\ell(u_{i,t}v_i) = a$ for $i = 1, 2, 3, 4, \ldots, n.$

Case 2: Assume that n is even and t is odd. In this case the labeling is exactly similar to Case 1.

Case 3: If n is odd and t is odd, define $\ell : E(W_0(t, n)) \to V_4 \setminus \{0\}$ as follows:

$$\ell(u_{i,1}u_{(i+1)(\text{mod } n),1}) = a \text{ for } i = 1, 2, 3, \ldots, n$$

for $j = 2, 3, \ldots, t$:
$\ell(u_{i,j}u_{i+1,j}) = b$ for $i = 1, 3, \ldots, n-2$
$\ell(u_{i,j}u_{(i+1)(\text{mod } n),j}) = c$ for $i = 2, 4, \ldots, n-1$
end for
$\ell(u_{n,i}u_{1,i}) = a$ for $i = 1, 2, 3, 4, \ldots, n,$
$\ell(u_{i,t}v_i) = a$ for $i = 1, 2, 3, 4, \ldots, n,$
for $k = 2, 3, \ldots, n-1$:
$\ell(u_{k,i}u_{k,i+1}) = a,$ for $i = 1, 2, 3, \ldots, t-1,$
end for
$\ell(u_{n,j}u_{1,j}) = a,$ for $j = 1, 2, 3, \ldots, t-1,$
$\ell(u_{1,i}u_{1,i+1}) = a,$ for $i = 1, 3, \ldots, t-2,$
$\ell(u_{1,i}u_{1,i+1}) = c,$ for $i = 2, 4, \ldots, t-1,$
$\ell(u_{n,i}u_{n,i+1}) = a,$ for $i = 1, 3, \ldots, t-2,$
$\ell(u_{n,i}u_{n,i+1}) = b,$ for $i = 2, 4, \ldots, t-1.$

Obviously $\ell^+(u_{i,j}) = a$ and $\ell^+(v_i) = a$.

□

Theorem 2.47. *$W_0(t, n) \notin \mathscr{V}_0$ for any n and t.*

Proof. Obvious. □

Definition 2.13. A closed helm $H(2, n)$ is the graph obtained from a helm by joining each pendant vertex to form a cycle [6]. A closed helm $H(2, n)$ is shown in figure 14.

Lemma 2.10. *If $\ell : E(H(2, n)) \to V_4 \setminus \{0\}$ is a labeling of $H(2, n)$, then*

$$\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(w) \tag{2.25}$$

where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \ldots, v_n$ are the vertices of the cycle $C_{n,2}$ and w is the central vertex.

Proof. Obvious. □

Theorem 2.48. *$H(2, n) \notin \mathscr{V}_a$ for any n.*

Proof. Assume that $H(2, n) \in \mathscr{V}_a$. Then by lemma 2.10, we have $na + na = a$. This implies that $a = 0$. This is a contradiction. □

Theorem 2.49. *$H(2, n) \in \mathscr{V}_0$ for all n.*

Proof. **Case 1** Assume that n is even. Define a labeling $\ell : E(H(2, n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} \ell(u_i w) &= a \quad \text{for} \quad i = 1, 2, \ldots, n, \\ \ell(u_i u_{i+1}) &= a \quad \text{for} \quad i = 1, 2, \ldots, n, \\ \ell(u_i v_i) &= a \quad \text{for} \quad i = 1, 2, \ldots, n, \\ \ell(v_i v_{i+1}) &= b \quad \text{for} \quad i = 1, 3, \ldots, n-1, \\ \ell(v_i v_{i+1}) &= c \quad \text{for} \quad i = 2, 4, \ldots, n. \end{aligned}$$

Obviously, ℓ is a zero- sum V_4 magic labeling of $H(2, n)$.

Case 2: Assume that n is odd. Define a labeling $\ell : E(H(2,n)) \to V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(u_1w) &= a,\ \ell(u_2w) = b,\ \ell(u_3w) = c,\\
\ell(u_iw) &= a, \quad \text{for} \quad i = 4,5,\dots,n,\\
\ell(u_iu_{i+1}) &= a \quad \text{for} \quad i = 1,2,\dots,n,\\
\ell(u_1v_1) &= a,\ \ell(u_2v_2) = b,\ \ell(u_3v_3) = c,\\
\ell(u_iv_i) &= a, \quad \text{for} \quad i = 4,5,\dots,n,\\
\ell(v_1v_2) &= c,\ \ell(v_2v_3) = a,\ \ell(v_3v_4) = b,\\
\ell(v_iv_{i+1}) &= c \quad \text{for} \quad i = 4,6,\dots,n-1,\\
\ell(v_iv_{i+1}) &= b \quad \text{for} \quad i = 5,7,\dots,n.
\end{aligned}$$

One can easily verify that ℓ is a zero sum magic labeling of $H(2,n)$.

□

Definition 2.14. Closed generalized helms $H(t,n)$ are obtained by taking a generalized web and joining pendent vertices to form a cycle [6].

Lemma 2.11. *If $\ell : E(H(t,n)) \to V_4 \setminus \{0\}$ is a labeling of $H(t,n)$, then*

$$\sum_{j=1}^{t}\sum_{i=1}^{n} \ell^+(u_{i,j}) = \ell^+(w). \tag{2.26}$$

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, and w is the central vertex.

Proof. Obvious. □

Theorem 2.50. *$H(t,n) \in \mathscr{V}_a$ if and only if both n and t are odd.*

Proof. First, assume that $H(t,n) \in \mathscr{V}_a$. Then by lemma 2.11, we have $(nt+1)a = 0$. This implies that both n and t are odd. Conversely, assume that both n and t are odd. Define $\ell : E(H(t,n)) \to V_4 \setminus \{0\}$ by:

$\ell(u_{1,1}w) = a,$
$\ell(u_{i,1}w) = b, \quad$ for $\quad i = 2,3,\dots,n,$
for $j = 1,2,\dots,t-1:$
$\quad \ell(u_{i,j}u_{i+1,j}) = c, \quad$ for $\quad i = 1,3,\dots,n-2,$
$\quad \ell(u_{i,j}u_{i+1,j}) = b, \quad$ for $\quad i = 2,4,\dots,n-1,n$
end for
$\ell(u_{i,t}u_{i+1,t}) = b, \quad$ for $\quad i = 1,3,\dots,n,$
$\ell(u_{i,t}u_{i+1,t}) = a, \quad$ for $\quad i = 2,4,\dots,n-1,$
for $j = 1,2,\dots,t-1:$
$\quad \ell(u_{1,j}u_{1,j+1}) = a,$
$\quad \ell(u_{i,j}u_{i,j+1}) = b, \quad$ for $\quad i = 2,3,\dots,n-1,$
end for
$\ell(u_{n,j}u_{n,j+1}) = c, \quad$ for $\quad j = 1,3,\dots,t-2,$
$\ell(u_{n,j}u_{n,j+1}) = b, \quad$ for $\quad j = 2,4,\dots,t-1.$

Obviously ℓ is an a-sum magic labeling of $H(t,n)$. □

Theorem 2.51. $H(t, n) \in \mathscr{V}_0$ *for all* n *and* t.

Proof. **Case 1:** Assume that n is even. Define $\ell : E(H(t, n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\ell(u_{i,1}w) = a, \quad \text{for} \quad i = 1, 2, \ldots, n. \\
&\text{for} \quad j = 1, 2, \ldots, t-1: \\
&\qquad \ell(u_{i,j}u_{i,j+1}) = a, \quad \text{for} \quad i = 1, 2, \ldots, n, \\
&\qquad \ell(u_{i,j}u_{i+1,j}) = a, \quad \text{for} \quad i = 1, 2, \ldots, n, \\
&\text{end for} \\
&\ell(u_{i,t}u_{i+1,t}) = b, \quad \text{for} \quad i = 1, 3, \ldots, n-1, \\
&\ell(u_{i,t}u_{i+1,t}) = c, \quad \text{for} \quad i = 2, 4, \ldots, n,
\end{aligned}$$

Obviously, ℓ is a zero sum V_4 magic labeling of $E(H(t, n))$.

Case 2: Assume that n is odd. Define $\ell : E(H(t, n)) \to V_4 \setminus \{0\}$ by:

$$\begin{aligned}
&\ell(u_{1,1}w) = a, \quad \ell(u_{2,1}w) = b \quad \ell(u_{3,1}w) = c \\
&\ell(u_{i,1}w) = a, \quad \text{for} \quad i = 4, 5, \ldots, n, \\
&\text{for} \quad j = 1, 2, \ldots, t-1: \\
&\qquad \ell(u_{i,j}u_{i+1,j}) = a, \quad \text{for} \quad i = 1, 2, \ldots, n, \\
&\qquad \ell(u_{1,j}u_{1,j+1}) = a, \\
&\qquad \ell(u_{2,j}u_{2,j+1}) = b, \\
&\qquad \ell(u_{3,j}u_{3,j+1}) = c, \\
&\qquad \ell(u_{i,j}u_{i,j+1}) = a, \quad \text{for} \quad i = 4, 5, , \ldots, n, \\
&\text{end for} \\
&\ell(u_{1,t}u_{2,t}) = c, \\
&\ell(u_{2,t}u_{3,t}) = a, \\
&\ell(u_{3,t}u_{4,t}) = b, \\
&\ell(u_{i,t}u_{i+1,t}) = c, \quad \text{for} \quad i = 4, 6, , \ldots, n-1, \\
&\ell(u_{i,t}u_{i+1,t}) = b, \quad \text{for} \quad i = 5, 7, , \ldots, n.
\end{aligned}$$

Obviously ℓ is a zero sum magic labeling of $E(H(t, n))$.

□

Theorem 2.52. $H(t, n) \in \mathscr{V}_{a,0}$ *if and only if both* n *and* t *are odd.*

Proof. Proof follows from 2.50 amd 2.51. □

Definition 2.15. The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to a central vertex of the helm [6](see figure 19).

Lemma 2.12. *If* $\ell : E(Fl_n)) \to V_4 \setminus \{0\}$ *is a labeling of* Fl_n*, then*

$$\sum_{i=1}^{n} \ell^+(u_i) + \sum_{i=1}^{n} \ell^+(v_i) = \ell^+(w) \tag{2.27}$$

Proof. Obvious. □

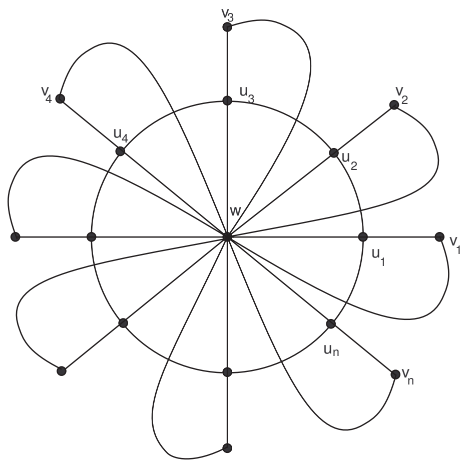

Figure 19: The flower graph: Fl_n

Theorem 2.53. $Fl_n \notin \mathscr{V}_a$ *for any* n.

Proof. Suppose $Fl_n \in \mathscr{V}_a$. Then by lemma 2.12, we hvae $na + na = a$. This implies that $a = 0$. This is a contradiction. ☐

Theorem 2.54. $Fl_n \in \mathscr{V}_0$ *for all* n.

Proof. If we label all the edges by a, we obtain that, $\ell^+(u_i) = 0, \quad \ell^+(v_i) = 0$ and $\ell^+(w) = 0.$ ☐

3 Conclusion

Let $V_4 = \{0, a, b, c\}$ be the Klein 4-group. In this paper, we identified a class of wheel related graphs in the following categories:

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

Acknowledgment

The authors thanks the anonymous referees and the editors for their valuable comments and suggestions on the improvement of this paper. Moreover, the first author is grateful to Kerala State Council for Science, Technology and Environment for the KSCSTE Fellowship under which this research was conducted.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Chartrand G, Zhang P. Introduction to graph theory. McGraw-Hill, Boston; 2005.

[2] Lee SM, Saba F, Salehi E, Sun H. On the V_4 magic graphs. Congressus Numerantium. 2002;156:59-67.

[3] Romain Boulet. Spectral characterizations of sun graphs and broken sun graphs. Discrete Mathematics and Theoretical Computer Science. 2009;11(2):149-160.

[4] Hafiz Muhammad Afzal Siddiqui, Muhammad Imran. Computing the metric dimension of wheel related graphs. Applied Mathematics and Computation. 2014;242:624-632.

[5] Hedge SM, Sudhakar Shetty. On arithmetic graphs. Indian Journal of pure applied mathematics. 2002;33(8):1275-1283.

[6] Seoud MA, Youssef MZ. Harmonious labellings of helms and related graphs, (unpublished).

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=1032&id=6&aid=8602