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Optimal Combinations of Control Strategies for Dynamics of Endemic Malaria Disease Transmission

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this study, a non-linear system of ordinary differential equation model that describe the dynamics of malaria disease transmission is derived and analyzed. Conditions are derived from the existence of disease-free and endemic equilibria. Basic reproduction number R_0 of the model is obtained, and we investigated that it is the threshold parameter between the extinction and persistence of the disease. If R_0 is less than unity, then the disease-free equilibrium point is both locally and globally asymptotically stable resulting in the disease removing out of the host populations. The disease can persist whenever R_0 is greater than unity. At R_0 is equal to unity, existence conditions are derived from the endemic equilibrium for both forward and backward bifurcations. Furthermore, optimal combinations of time dependent control measures are incorporated to the model, and we derived the necessary conditions of the optimal control using Pontryagins's maximum principal theory. Numerical simulations were conducted using MATLAB software to confirm our analytical results. Our findings were that malaria disease may be controlled more with strict application of the combination of all control measures that is, the combination of prevention of drug resistance, insecticide treated net ITN, indoor residual spray IRS and active treatment than when the combination of three control measures are used.

Keywords: Malaria; disease-free equilibrium; endemic equilibrium; basic reproduction number; stable; optimal control.

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1 Introduction

Malaria is an old infectious parasitic disease and transmitted to human through the bites of infected female Anopheles' mosquitoes [1]. "The burden of malaria disease affects the community socioeconomic in many ways. Some of these are fertility, population growth, saving and investment, worker productivity, absenteeism, premature mortality and medical costs" [2]. "In areas where malaria is highly endemic, young children bears a larger burden in terms of the disease morbidity and mortality and affects fetal development during early stage of pregnancy in women due to loss of immunity. Malaria is still treating a serious challenge to the global world population. According to 2020 world health organization (WHO) report, 241 million cases and 627 thousand deaths from malaria globally and the estimate number of children under 5 years of age deaths caused by malaria only in Africa is 80%" [3].

The most popular strategies of controlling malaria disease includes, the use of chemotherapy, intermittent preventive treatment for children and pregnant women (preventive doses of sulfadoxine pyrimethamine (IPT/ST)), and use of insecticides treated bed nets and insecticides against the vector. The challenge posed by the resistance of parasites against drugs and resistance of mosquitoes against insecticides calls for urgent need for a better understanding of important parameters in the disease transmission and develops effective and optimal strategies for prevention and control of the spread of malaria disease

"Mathematical modeling has become an important tool in understanding the complex dynamics of disease transmission and in decision making processes regarding intervention programs for disease control. Concerning malaria disease, Ross (1911) developed the first mathematical model. He focused his study on mosquito control and showed that for the disease to be eliminated the mosquito population should be brought below a certain threshold" [4]. Later the idea of Ross is extended by Macdonald to account for super infection [5]. Ngwa, G. A. Shu, W.S., A mathematical model for endemic malaria with variable human and mosquito population [6]. Alemu G. W., Boka K.B.,, P.R. Koya derived and analyzed deterministic model for the inclusion of Infected immigrants on the spread and dynamics of malaria transmission [7], Chiyaka,C., Garira, and W., Dube, S., derived analyzed effects of treatment and drug resistance on the transmission dynamics of malaria in endemic areas [8], J.Tumwiine, S.D.H-Musekwa and F.Nyabadaza were analyzed a mathematical model for the transmission and spread of drug sensitive and resistant malaria strains within human populations [9]. Other studies are carried out by using optimal control theory. Okosun *et al*. derived and analyzed a malaria disease transmission mathematical model that includes treatment and vaccination with waning immunity and applied optimal control to study the impact of a possible vaccination with treatment strategies in controlling the spread of malaria [10], F.B. Agusto, and M.A. Khan, derived and analyzed Optimal Control Strategies for dengue transmission [11], K. O. Okosun and O. D. Makinde Modelling the impact of drug resistance in malaria transmission and its optimal control analysis [12], E. Bonyah, M.A. Khan,K.O. Okosun, J.F. Gómez-Aguilar present ''Modeling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhea with optimal [13]. Makinde and Okosun, were applied optimal control to study the impact of chemo-therapy on malaria disease with infective immigrants [14], K. O. Okosun, O. Rachid, and N. Marcus, applied optimal control strategies and cost-effectiveness analysis of a malaria model [15]. Temesgen D. K,, O. D.Makinde & Legesse L. O. derived and analyzed Optimal Control and Cost Effectiveness Analysis of SIRS Malaria Disease Model with Temperature Variability[16].

In this paper, we study SITRS-SI and SIRS-SI endemic malaria transmission model with standard incidence law that was presented by [12]. Furthermore, we modified the model [6] by omitting the incubating class from the system and incorporate four time dependent control measures, the class infective in treatment individuals and infectious classes with drug sensitive and drug resistant individuals. The purpose of this study is

(i) to investigate the stability for both disease-free equilibrium and endemic equilibrium
(ii) to develop effective ways for controlling the malaria disease
(iii) to explore the best strategy in terms of reducing the number of malaria infectious populations to zero.

2 Model Description and Formulation

The model subdivides the human populations in to five sub class namely, susceptible $S_{\square}$, infected with drug sensitive malaria strain I_{hs}, infected with drug resistant malaria strain I_{hr}, infective in treatment $T_{\square}$, recovered

$R_{\square}$. Similarly, the mosquito populations are also sub divided in to susceptible class S_v, and infected class I_v.The total number of human and mosquito populations at time t are denoted and given by $N_{\square}(t) = S_{\square}(t) + I_{hs}(t) + I_{hr}(t) + T_{\square}(t) + R_{\square}(t)$ and $N_v(t) = S_v(t) + I_v(t)$ respectively. Note that, $S_{\square} = S_{\square}(t)$, $I_{hs} = I_{hs}(t)$, $I_{hr} = I_{hr}(t)$, $T_{\square} = T_{\square}(t)$, $R_{\square} = R_{\square}(t)$, $S_v = S_v(t)$, $I_v = I_v(t)$ and $N_{\square} = N_{\square}(t)$. The susceptible humans S_h are recruited at the rate of Λ_h and they either die from natural causes at a rate of μ_h or move to infected class I_h by acquiring malaria through contact with infectious mosquitoes with respective rate of force of infection $\lambda_h = \phi\omega\beta_h \frac{I_v}{N_h}$. Where, β_h is the rate of probability of human getting infected, ϕ is the mosquito contact rate with human and ω is mosquito biting rate. We also let a fraction ρ of humans be infected with drug sensitive malaria strain and the remaining fraction $(1-\rho)$ individuals are infected with drug resistant malaria strains. Infected humans with drug sensitive malaria strains I_{hs} individuals are either die from natural causes and due to disease death with respective rates μ_h and δ_h respectively or move to infective in treatment T_h at a rate α and recovered class R_h at recovery rate γ_s. Infected with drug resistant malaria strains I_{hr} individuals are also either die from natural causes and due to disease death with respective rates μ_h and δ_h respectively or move to infective in treatment T_h at a rate σ and recovered class R_h at recovery rate γ_r. Infective in treatment T_h individuals are individuals with malaria disease that are getting treated under the control. They also either die from natural causes and due to disease death with respective rates μ_h and δ_h respectively or move to the susceptible class with fraction of ε due to the administered drug kills off the parasites and the infected humans with drug resistant malaria strains class with fraction of π due to treatment failure. These infected with drug sensitive and resistant malaria strains individuals progress to partially immune group (recovered class).Partially immune group(recovered individuals) either losses immunity and becomes again move to susceptible class with respective rate θ or die from natural death at a rate $\mu_{\square}$.Susceptible mosquitoes S_v are recruited at the rate Λ_v. They either die due to natural death at a rate of μ_v or move to infected class I_v by acquiring malaria through contact with infectious humans with respective rate of force of infection $\lambda_v = \phi\omega\beta_v \left(\frac{I_{hs}+I_{hr}}{N_{\square}}\right)$. Where β_v is the probability of a mosquito getting infected. Infected mosquitoes I_v are die because of natural and disease induced death with respective rates μ_v and δ_v respectively. No recovered compartment for mosquitoes. We represent diagrammatically the flow of both the human and mosquito populations from one class to the other is given below.

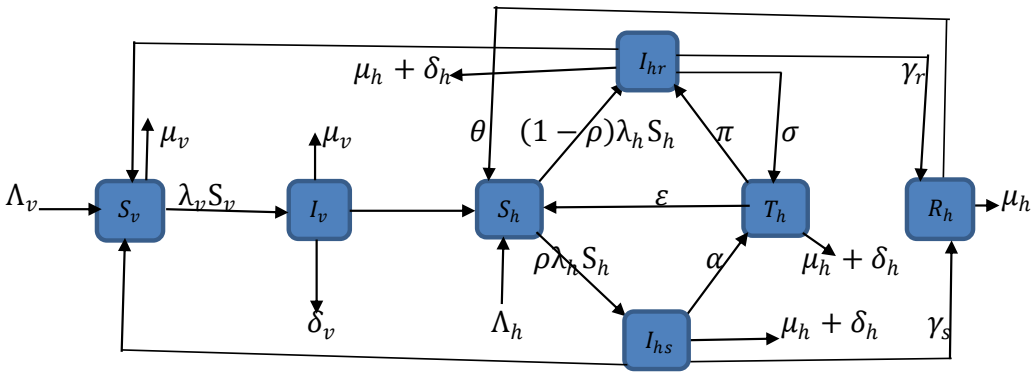


Fig. 1. Flow diagram for the transmission of endemic malaria model

$$\begin{cases} \frac{dS_h}{dt} = \Lambda_h + \varepsilon T_h + \theta R_h - (\lambda_h + \mu_h)S_h \\ \frac{dI_{hs}}{dt} = \rho(1-\pi)\lambda_h S_h - (\mu_h + \delta_h + \alpha + \gamma_s)I_{hs} \\ \frac{dI_{hr}}{dt} = (1-(1-\pi)\rho)\lambda_h S_h - (\mu_h + \delta_h + \gamma_r + \sigma)I_{hr} \\ \frac{dT_h}{dt} = \alpha I_{hs} + \sigma I_{hr} - (\delta_h + \mu_h + \varepsilon)T_h \\ \frac{dR_h}{dt} = \gamma_s I_{hs} + \gamma_r I_{hr} - (\theta + \mu_h)R_h \\ \frac{dS_v}{dt} = \Lambda_v - (\lambda_v + \mu_v)S_v \\ \frac{dI_v}{dt} = \lambda_v S_v - (\mu_v + \delta_v)I_v \end{cases} \quad (2.1)$$

With initial conditions

$$S_h(0) = S_{0\square}, I_{hs}(0) = I_{0\square s}, I_{hr}(0) = I_{0\square r}, T_h(0) = T_{0\square}, R_h(0) = R_{0\square}, S_v(0) = S_{0v}, I_v(0) = I_{0v} \quad (2.2)$$

With some of the following additional assumptions

(i) The susceptible class in both the human and mosquito populations enter into the infective classes by adequate contact with infectious populations not infective in treatment.
(ii) infective in treatment and recovered individuals are not infectious to the susceptible populations
(iii) Those infective humans recovered from the disease due to natural immunity and enter into a partially immune group
(iv) Those infective individuals in treatment recovered from the disease due to the administered drug killed off the parasites
(v) One part of the recovered class again becomes susceptible to the disease
(vi) No recovered compartment for mosquitoes.

3 Basic Property of the Model

3.1 Positivity of the model

Theorem 1 Every solution of system (2.1) with initial conditions equation (2.2) exists in the

Interval $[0,\infty)$ and $S_h(t) \geq 0, I_{hs}(t) \geq 0, I_{hr}(t) \geq 0. T_h(t) \geq 0, R_h(t) \geq 0$, $S_v(t) \geq 0$ and $I_v(t) \geq 0$ for all $t \geq 0$.

Proof. To show positivity of solutions, it is enough to show that each of the trajectories of system (2.1) is non-negative for all $t \geq 0$.

Proof. To show positivity of solutions, it is enough to show that each of the trajectories of system (2.1) is non-negative for all $t \geq 0$.

Since the right-hand side of system (2.1) is completely continuous and locally Lipschitzian on C, the solution $\left(S_{\square}(t),\ I_{hs}(t),\ I_{hr}(t), T_{\square}(t), R_{\square}(t), S_v(t),\ I_v(t)\right)$ of system (2.1) with initial condition equation (2.2) exists and unique on [0, k) where $0 < k < +\infty$.

It follows from the first system (2.1) that, the differential inequality describing the evolution of the susceptible human population over time t is given by

$$\frac{dS_h}{dt} \geq \Lambda_h - \left(\phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(t) + \mu_h\right) S_{\square}(t)$$
$$\frac{d}{dt}\left[S_{\square}(t) exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}\right] \geq \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}$$

Hence,

$$S_{\square}(t) exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\} - S_{0\square} \geq \int_{\tilde{t}}^t \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(\psi) d\psi\right\} dt$$

Thus,

$$S_{\square}(t) \geq S_{0\square} exp\left[-\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(S_{\square})\, ds\right\}\right] + exp\left[-\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}\right] \times$$
$$\int_0^t \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(\psi) d\psi\right\} dt > 0.$$

From the second system (2.1) we have,

$\frac{dI_{hs}}{dt} \geq -(\mu_h + \delta_h + \alpha + \gamma_s)I_{hs}(t)$ is equivalent to $I_{hs}(t) \geq exp[-(\mu_h + \delta_h + \alpha + \gamma_s)t] > 0.$

From the third system (2.1) we have,

$\frac{dI_{hr}}{dt} \geq -(\mu_h + \delta_h + \gamma_r + \sigma)I_{hr}(t)$ is equivalent to $T_h(t) \geq exp[-(\mu_h + \delta_h + \gamma_r + \sigma)t] > 0.$

From the fourth system (2.1) we have,

$\frac{dT_h}{dt} \geq -(\delta_h + \mu_h + \varepsilon)T_h(t)$ is equivalent to $T_h(t) \geq exp[-(\mu_h + \delta_h + \varepsilon)t] > 0.$

From the fifth system (2.1) we have,

$\frac{dR_h}{dt} \geq -(\mu_h + \theta)I_h(t)$ is equivalent to $R_h(t) \geq exp[-(\mu_h + \theta)t] > 0.$

From the sixth system (2.1) we have,

$$\frac{dS_v}{dt} \geq \Lambda_v - \left(\int_0^t \phi\omega\beta_v \left(\frac{I_{hs} + I_{hr}}{N_h}\right)(t) + \mu_v \right) S_v$$

$$\frac{d}{dt}\left[S_v(t) exp\left\{ \mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs} + I_{hr}}{N_h}\right)(s)\, ds \right\} \right] \geq \Lambda_v exp\left\{ \mu_h t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs} + I_{hr}}{N_h}\right)(s)\, ds \right\}$$

Hence,

$$S_v(t) exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs}+I_{hr}}{N_h}\right)(s)\, ds\right\} - S_{0v} \geq \int_{\tilde{t}}^t \Lambda_v exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs}+I_{hr}}{N_h}\right)(\psi) d\psi\right\} dt$$

Thus,

$$S_v(t) \geq S_{0\square} exp\left[-\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs}+I_{hr}}{N_h}\right)(s)\, ds\right\}\right] + exp\left[-\left\{\mu_v t + \int_0^t \phi\omega\beta_h \left(\frac{I_{hs}+I_{hr}}{N_h}\right)(s)\, ds\right\}\right] \times$$
$$\int_0^t \Lambda_v exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_{hs}+I_{hr}}{N_h}\right)(\psi) d\psi\right\} dt > 0.$$

From the seventh system (2.1) we have,

$\frac{dI_v}{dt} \geq -(\mu_v + \delta_v)I_v(t)$ is equivalent to $I_v(t) \geq exp[-(\mu_v + \delta_v)t] > 0.$

Therefore; we can see that $S_{\square}(t) > 0$, $I_{hs}(t) > 0$, $I_{hr}(t) > 0, T_{\square}(t) > 0, R_{\square}(t) > 0, S_v(t) > 0$, $I_v(t) > 0$ for all $t \geq 0$.

3.2 Invariant region

Theorem 2 The feasible region Γ defined by

$\Omega = \{\Omega_h \times \Omega_v\} \subset \{\square_+^5 \times \square_+^2\}$ where, $\Omega_h = \left\{(S_{\square} \quad I_{hs}, \quad I_{hr}, \quad T_{\square}, \quad R_{\square}) \in \square_+^5 : N_h \leq \frac{\Lambda_h}{\mu_h}\right\}$ and

$\Omega_v = \left\{(S_v, \quad I_v,) \in \square_+^2 : N_v \leq \frac{\Lambda_v}{\mu_v}\right\}$, with initial conditions $S_h(0) = S_{0\square}$, $I_{hs}(0) = I_{0\square s}$, $T_h(0) = T_{0\square}$, $R_h(0) = R_{0\square}$, $S_v(0) = S_{0v}$, $I_v(0) = I_{0v}$, is bounded.

Proof: Let $N_{\square}(t) = S_{\square}(t) + I_{hs}(t) + I_{hr}(t) + T_{\square}(t) + R_{\square}(t)$ and $N_v(t) = S_v(t) + I_v(t)$

The feasible region of both the human and mosquito populations are determined by the feasible region of $N_{\square}(t)$ and $N_v(t)$ respectively as follows

The feasible region of $\boldsymbol{N_h(t)}$**:** Total sum of human compartments of system (2.1) leads to

$\frac{dN_{\square}}{dt} = \Lambda_{\mathrm{h}} - \mu_{\square}N_{\square}(t) - \delta_{\square}(I_{hs}(t) + I_{hr}(t) + T_{\square}(t))$ if and only if
$\frac{dN_{\square}}{dt} \leq \Lambda_{\mathrm{h}} - \mu_{\square}N_{\square}(t)$ if and only if
$\frac{dN_{\square}}{dt} + \mu_{\square}N_{\square}(t) \leq \Lambda_{\mathrm{h}}$

The resulting differential inequality can be solved by separation of variables to give,

$\int \frac{d}{dt}(N_{\square}e^{\mu_{\square}t}) \leq \int \Lambda_{\mathrm{h}}e^{\mu_{\square}t}$
$\int \frac{d}{dt}(N_{\square}e^{\mu_{\square}t}) \leq \int \Lambda_{\mathrm{h}}e^{\mu_{\square}t}$

Taking the initial conditions $t = 0$ and denoting $N_h(0)$ by $N_{0\square}$, then the complete solution

$$N_{\square}(t) \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}} + \left(N_{0\square} - \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}\right)e^{-\mu_{\mathrm{h}}\mathrm{t}}.$$

As $t \to \infty, 0 < N_{\square} \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$ .So if $N_{0\square} \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$, then $\lim_{t\to\infty} N_{\square}(t) \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$ .This means that $\frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$ is upper bound of N_h.On the other hand if $N_{0h} > \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$,then $N_h(t)$ will decrease to $\frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$ Thus $N_{0h} \leq N_h(t) \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}$. Therefore; the total human population is bounded.

The feasible region of $\boldsymbol{N_v(t)}$**:** total sum of mosquito compartments of the system (2.1)

leads to $\frac{dN_v}{dt} = \Lambda_v - \mu_v N_v - \delta_v I_v$
.
$\frac{dN_v}{dt} \leq \Lambda_v - \mu_v N_v(t)$
$\frac{dN_v}{dt} + \mu_v N_v(t) \leq \Lambda_v$

The resulting differential inequality can be solved by separation of variables to give,

$$\int \frac{d}{dt}(N_v e^{\mu_v t}) \leq \int \Lambda_{\mathrm{v}} e^{\mu_v t}$$

Taking the initial conditions $t = 0$ and denoting $N_v(0)$ by N_{0v}, then the complete solution

$$N_v(t) \leq \frac{\Lambda_{\mathrm{v}}}{\mu_v} + \left(N_{0v} - \frac{\Lambda_{\mathrm{v}}}{\mu_v}\right)exp(-\mu_v \mathrm{t}).$$

As $t \to \infty, 0 < N_v \leq \frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$ .So if $N_{0v} \leq \frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$, then $\lim_{t\to\infty} N_v(t) \leq \frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$ .This means that $\frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$ is upper bound of N_v.On the other hand if $N_{0v} > \frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$,then $N_v(t)$ will decrease to $\frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$.Thus $N_{0v} \leq N_v(t) \leq \frac{\Lambda_{\mathrm{v}}}{\mu_{\mathrm{v}}}$.

Therefore; the total mosquito population is bounded. Thus, the solutions of the model variables representing human populations $\{(S_h, \quad I_h, \quad T_h, \quad R_h)\}$

are confined in the feasible region $\Omega_{\mathrm{h}} = \left\{(S_h(t), \quad I_{hs}(t), I_{hr}(t), \quad T_h(t), \quad R_h(t)) \in \mathbb{R}_+^5 : \mathrm{N_h} \leq \frac{\Lambda_{\mathrm{h}}}{\mu_{\mathrm{h}}}\right\}$. Similarly, the solutions of the model variables representing mosquito populations $\{(S_v, \quad I_v)\}$ are confined in the feasible region $\Omega_{\mathrm{v}} = \left\{(\mathrm{S_v}, \quad I_v) \in \mathbb{R}_+^2 : \mathrm{N_v} \leq \frac{\Lambda_{\mathrm{v}}}{\mu_v}\right\}$.

This shows that the feasible region of the system system (2.1) is bounded and is given by $\Omega = \{S_h(t),\ I_{hs}(t), I_{hr}(t),\ T_h(t),\ R_h(t),\ S_v(t),\ I_v(t)\} \in \mathbb{R}_+^7$ or equivalent to $\Omega = \{\Omega_h \times \Omega_v\} \subset \{\mathbb{R}_+^5 \times \mathbb{R}_+^2\}$.

Thus, in Ω the system (2.1) is well-posed epidemiologically and mathematically. Hence, it is sufficient to study the dynamics of the model in Ω.

4 Disease-free Equilibrium and Basic Reproduction Number, Disease-free Stability

The disease-free equilibrium point of the model is its steady state solutions without infection or disease.

Consider the disease free-equilibrium point denoted and given by:

$$E_0 = (S_h^0 \quad I_{hs}^0 \quad I_{hr}^0 \quad T_h^0 \quad R_h^0 \quad S_v^0 \quad I_v^0)$$

Where, S_h^0 , I_{hs}^0, , I_{hr}^0, T_h^0 , R_h^0 , S_v^0 and I_v^0 are the components of E_0 and $I_{hs}^0 = I_{hr}^0 = T_h^0 = R_h^0 = I_v^0 = 0$ and the non-infectious are obtained by setting $\frac{dS_h}{dt} = \frac{dS_v}{dt} = 0$ for the malaria model system (2.1) and after computing the resultant gives $S_h^0 = \frac{\Lambda_h}{\mu_h}$, and $S_v^0 = \frac{\Lambda_v}{\mu_v}$ Hence;

$$E_0 = \left(\frac{\Lambda_h}{\mu_h} \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{\Lambda_v}{\mu_v} \quad 0\right) \tag{4.1}$$

The basic reproduction number denoted by R_0 is the average number of secondary infectious infected by an infective individual during his or her whole course of disease [17]. We use the next generation matrix method by van den Driessche and Watmough [18] to derive the basic reproduction number R_0 of system (2.1). The infectious compartment of system (2.1) are, I_{hs}, I_{hr}, and I_v. To apply the method [18], let the system (2.1) be rearranged by beginning with the infected classes as follows:

Let $X = (I_{hs} \quad I_{hr} \quad T_h \quad I_v \quad S_h \quad R_h \quad S_v)^T$

$$F(X_i) = \begin{pmatrix} \rho(1-\pi)\frac{\phi\omega\beta_h I_v}{N_h} S_h \\ (1-(1-\pi)\rho)\frac{\phi\omega\beta_h I_v}{N_h} S_h \\ \frac{\phi\omega\beta_v (I_{hs}+I_{hr})}{N_h} S_v \end{pmatrix} \text{ and } V(X_i) = \begin{pmatrix} (\mu_h + \delta_h + \alpha + \gamma_s) I_{hs} \\ (\mu_h + \delta_h + \sigma + \gamma_r) I_{hr} \\ (\mu_v + \delta_v) I_v \end{pmatrix}$$

The new infection matrix F and the transition matrix V are given, respectively, by

$$F\begin{pmatrix} 0 & 0 & (1-\pi)\rho\phi\omega\beta_h \\ 0 & 0 & (1-(1-\pi)\rho)\phi\omega\beta_h \\ \frac{\phi\omega\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} & \frac{\phi\omega\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} & 0 \end{pmatrix}$$

$$\text{and } V = \begin{pmatrix} (\mu_h + \delta_h + \alpha + \gamma_s) & 0 & 0 \\ 0 & (\mu_h + \delta_h + \sigma + \gamma_r) & 0 \\ 0 & 0 & (\mu_v + \delta_v) \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} 0 & 0 & \frac{(1-\pi)\rho\phi\omega\beta_h}{(\mu_v+\delta_v)} \\ 0 & 0 & \frac{(1-(1-\pi)\rho)\phi\omega\beta_h}{\mu_v+\delta_v} \\ \frac{\phi\omega\beta_v\mu_h\Lambda_v}{(\mu_h+\delta_h+\alpha+\gamma_s)\Lambda_h\mu_v} & \frac{\phi\omega\beta_v\Lambda_v\mu_h}{(\mu_h+\delta_h+\sigma+\gamma_r)\Lambda_h\mu_v} & 0 \end{pmatrix} \text{ and}$$

The basic reproduction number of system (2.1) is the dominant eigen value of the next generation matrix FV^{-1} which is given by

$$R_0 = \sqrt{\frac{\phi^2\omega^2\beta_h\beta_v\mu_h\Lambda_v\left((\mu_h+\delta_h+\sigma+\gamma_r)\rho(1-\pi)+(\mu_h+\delta_h+\alpha+\gamma_s)(1-(1-\pi)\rho)\right)}{\Lambda_h\mu_v(\mu_h+\delta_h+\alpha+\gamma_s)(\mu_h+\delta_h+\sigma+\gamma_r)(\mu_v+\delta_v)}} \tag{4.2}$$

4.1 Local stability of disease-free equilibrium point

Theorem3 The disease-free equilibrium point E_0 of system (2.1) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof:

The local stability of the system is determined by the signs of the eigenvalues and it is further proved by linearizing to obtain its Jacobian at disease-free steady-state points so that

The Jacobian matrix of system (2.1) at disease free equilibrium point E_0 defined and given by

$$J(E_0) = \begin{pmatrix} -\mu_h & 0 & 0 & \varepsilon & \theta & 0 & -m_{17} \\ 0 & -m_1 & 0 & 0 & 0 & 0 & m_{27} \\ 0 & 0 & -m_2 & 0 & 0 & 0 & m_{37} \\ 0 & \alpha & \sigma & -m_3 & 0 & 0 & 0 \\ 0 & \gamma_s & \gamma_r & 0 & -m_4 & 0 & 0 \\ 0 & -m_{62} & -m_{63} & 0 & 0 & -\mu_v & 0 \\ 0 & m_{72} & m_{73} & 0 & 0 & 0 & -m_5 \end{pmatrix} \tag{4.3}$$

Where, $m_1 = (\mu_h + \delta_h + \alpha + \gamma_s)$, $m_2 = (\mu_h + \delta_h + \sigma + \gamma_r)$, $m_3 = (\mu_h + \delta_h + \varepsilon)$, $m_4 = (\mu_h + \theta)$, $m_5 = (\mu_v + \delta_v)$, $m_{17} = \phi\omega\beta_h$, $m_{27} = (1-\pi)\rho\phi\omega\beta_h, m_{37} = (1-(1-\pi)\rho)\phi\omega\beta_h$ $m_{62} = m_{63} = m_{72} = m_{73} = \frac{\phi\omega\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v}$

$Det(J(E_0) - \lambda I) = 0$ if and only if $\lambda_1 = -\mu_h < 0, \lambda_2 = -\mu_v < 0,\ \lambda_3 = -m_2 < 0,$

$\lambda_4 = -m_4 < 0$ and

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{4.4}$$

Where,

$$\begin{aligned} a_0 &= 1, \\ a_1 &= m_1 + m_2 + m_5 \\ a_2 &= m_2m_5 + m_1(m_2 + m_5) - \frac{\phi^2\omega^2\beta_h\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} \\ a_3 &= m_1m_2m_5(1 - R_0^2) \end{aligned} \tag{4.5}$$

By the principle of Ruth-Hurwitz criteria [19], equation (4.3) has negative real eigenvalues if and only if $a_1 > 0, a_3 > 0$ and $a_1\ a_2 > a_3$. Clearly, we see that, $a_1 > 0$ because of it is the sum of positive variables, but $a_3 > 0$ if and only if $1 - R_0^2 > 0$ which is equivalent to $R_0 < 0$ and hence, all eigenvalues of the determinant of equation (4.3) will have negative real eigenvalues. Therefore; the disease-free equilibrium point E_0 is locally asymptotically stable.

5 Existence of Endemic Equilibrium and Bifurcation, and Local Stability of Endemic Equilibrium

Let $E^* = (S_h^*,\ I_{hs}^*,\ I_{hr}^*,\ T_h^*,\ R_h^*,\ S_v^*,\ I_v^*)$ be a non-trivial endemic equilibrium point of system (2.1), that is all components of E_* are positive. If we set system (2.1) to zero we get the following

$$S_h^* = \frac{m_1m_2m_3m_4\Lambda_h}{m_1m_2m_3m_4\mu_h + (\mu_h k_3 + k_4)\lambda_h^*},\ I_{hs}^* = \frac{\rho(1-\pi)\lambda_h^* S_h^*}{m_1} =,\quad I_{hr}^* = \frac{(1-(1-\pi)\rho)\lambda_h^* S_h^*}{m_2},\quad S_v^* = \frac{\Lambda_v}{\mu_v+\lambda_v^*},\ I_v^* = \frac{\Lambda_v\lambda_v^*}{(\mu_v+\lambda_v^*)m_5},\ T_h^* = \frac{(m_2\rho\alpha(1-\pi)+\sigma m_1(1-(1-\pi)\rho))\lambda_h^* S_h^*}{m_1m_2m_3},\ R_h^* = \frac{(\gamma_s(1-\pi)\rho m_2+\gamma_r(1-(1-\pi)\rho)m_1))\lambda_h^* S_h^*}{m_1m_2m_4} \quad (5.1)$$

$$\lambda_h^* = \phi\omega\beta_h \frac{I_v^*}{N_h^*} \quad (5.2)$$

$$\lambda_v^* = \phi\omega\beta_v \frac{(I_{hs}^* + I_{hr}^*)}{N_h^*} \quad (5.3)$$

Where, $N_h^* = S_h^* + I_{hs}^* + I_{hr}^* + T_h^* + R_h^*$ and substituting (5.1) in to (5.3) we get

$$\lambda_v^* = \frac{\phi\omega\beta_v((1-\pi)\rho m_2+(1-(1-\pi)\rho)m_1)m_1m_2m_3m_4\lambda_h^*}{m_1m_2(m_1m_2m_3m_4+k_3\lambda_h^*)} \quad (5.4)$$

Again substituting (5.1) and (5.4) respectively in to (5.2) we get

$$\lambda_h^*(b_0(\lambda_h^*)^2 + b_1\lambda_h^* + b_2) = 0 \quad (5.5)$$

Where,

$$\begin{aligned} b_0 &= k_3 m_1 m_2 m_3 m_4 m_5 \Lambda_h \left(\mu_v k_3 m_1 m_2 + \phi\omega\beta_v m_1 m_2 m_3 m_4\big((1-\pi)\rho m_2 + (1-(1-\pi)\rho)m_1\big)\right) \\ b_1 &= \frac{m_1 m_2 m_5 \mu_v \Lambda_h}{\mu_h}(R_c^2 - R_0^2) \\ b_2 &= m_1^4 m_2^4 m_3^3 m_4^3 m_5 \mu_v \Lambda_h (1 - R_0^2) \end{aligned} \quad (5.6)$$

Where, R_0 is the basic reproduction number given by (4.2) and

$$R_c = \sqrt{\frac{\mu_h m_3 m_4\left(\phi\omega\beta_v m_1 m_2 m_3 m_4\big((1-\pi)\rho m_2+(1-(1-\pi)\rho)m_1\big)+2\mu_v m_1 m_2 k_3\right)}{m_1^2 m_2^2 m_3^2 m_4^2 \mu_v(\mu_h k_3 + k_4)}}$$

$$k_1 = \alpha(1-\pi)\rho m_2 + \sigma(1-(1-\pi)\rho)m_1),\ k_2 = \gamma_s(1-\pi)\rho m_2 + \gamma_r(1-(1-\pi)\rho)m_1$$

$$k_3 = m_4 k_1 + m_3 k_2 + m_3 m_4\big((1-\pi)\rho m_2 + (1-(1-\pi)\rho)m_1\big),$$

$$k_4 = \delta_h m_4\big((1-\pi)\rho m_2(m_3+\alpha) + (1-(1-\pi)\rho)m_1(m_3+\sigma)\big)$$

Equation (5.5) admits a trivial solution $\lambda_h^* = 0$ which corresponds to the disease-free equilibrium point (DFEP). Now we assume $\lambda_h^* \neq 0$ the existence of endemic equilibria is regulated by the quadratic equation $b_0(\lambda_h^*)^2 + b_1\lambda_h^* + b_2 = 0$. The coefficient b_0 in (5.6) is always positive and b_2 is positive if $R_0 < 1$ and negative if $R_0 > 1$. So, the sign of b_1 and b_2 will decide about the positive solution of (5.5). For the case when $R_0 > 1$, two solutions can be obtained for (5.5), that are positive and negative. For the case when considering $b_2 = 0$ if and only if $R_0 = 1$, then a solution of the form $\lambda_h^* = \frac{-b_1}{b_0}$ exists when $b_1 < 0$ ($R_c < R_0$). It follows that the number of endemic equilibria of (2.1) is depend on the coefficient b_0, b_1 and b_2 as follows:

Theorem 4 The system (1) has

(i) a unique endemic equilibrium if $b_2 < 0$ if and only if $R_0 > 1$
(ii) a unique endemic equilibrium if $b_1 < 0$ and $b_2 = 0$ or ($b_1 < 0$, $b_2 > 0$ and $b_1^2 - 4b_0b_2 = 0$)
(iii) Two endemic equilibrium if $b_2 > 0$ and $b_1 < 0$ and $b_1^2 - 4b_0b_2 > 0$
(iv) otherwise no endemic equilibrium

Here also, when put for the value of $\Lambda_h = 0.071$ from [20] and use Table 1 for the values of other parameters, the two roots are presented graphically as shown in Fig. 2. Where, the blue line represents stable equilibrium and the red line represents unstable equilibrium.

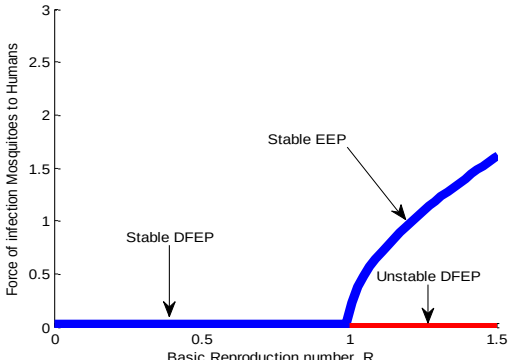


Fig. 2. When we plot the basic reproduction number R_0 versus the force of infection mosquitoes to humans, we note stable disease free region when $\lambda_h^* = 0$ and when $R_0 = 1$, the force of infection mosquitoes to humans starts to increase in stable endemic region where we note that the disease start to spread again and hence, forward bifurcation

5.1 Existence of back ward bifurcation

To show the existence of backward bifurcation of system (2.1), we employ the method developed in Gumel and Song, 2008; Castillo-Chavez and Song, 2004 [21-23]. We also assume and note that, the normal form representing the dynamics of the system on the Centre manifold theory is given by $\dot{\mu} = a\mu^2 + b\xi\mu$, where,

$$a = \frac{v}{2}.D_{xx}f(x_0,0)w^2 = \frac{1}{2}\sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(x_0,0) \neq 0 \text{ for j =1,2…,n} \tag{5.7}$$

$$b = V.D_{x\xi}f(x_0,0)w = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \xi}(x_0,0) \neq 0 \text{ for i =1,2…,n} \tag{5.8}$$

Where,

ξ Denotes a bifurcation parameter to be chosen,
f_ks Denote the right hand side of system(2.1),
x Denotes the state vector,
x_0 Denotes the disease-free equilibrium E_0,
D_x Denotes the differential operator with respect to x,
D_ξ Denotes the differential operator with respect to ξ, and
w, v denotes the right and left eigenvectors respectively corresponding to the null eigenvalue of the Jacobian matrix of (2.1), evaluated at x_0 for $\xi = 0$.

Let we choose the rate of transmission of infection from an infectious mosquito to a susceptible human β_h as the bifurcation parameter. We observe that $R_0 = 1$ is equivalent to:

$$\beta_h = \beta_h^* = \frac{\Lambda_h \mu_v m_1 m_2 m_5}{\phi^2 \omega^2 \beta_v \mu_h \Lambda_v (m_2\rho(1-\pi)+m_1(1-(1-\pi)\rho))} \tag{5.9}$$

and the linearized Jacobian matrix evaluated at E_0 and β_h^* denoted and given by

$$J(E_0,\beta_h^*) = \begin{pmatrix} -\mu_h & 0 & 0 & \varepsilon & \theta & 0 & -m_{17}^* \\ 0 & -m_1 & 0 & 0 & 0 & 0 & m_{27}^* \\ 0 & 0 & -m_2 & 0 & 0 & 0 & m_{37}^* \\ 0 & \alpha & \sigma & -m_3 & 0 & 0 & 0 \\ 0 & \gamma_s & \gamma_r & 0 & -m_4 & 0 & 0 \\ 0 & -m_{62} & -m_{63} & 0 & 0 & -\mu_v & 0 \\ 0 & m_{72} & m_{73} & 0 & 0 & 0 & -m_5 \end{pmatrix} \tag{5.10}$$

$m_{17}^* = \phi\omega\beta_h^*, \quad m_{27}^* = (1-\pi)\rho\phi\omega\beta_h^*, m_{37}^* = (1-(1-\pi)\rho)\phi\omega\beta_h^*$
$Det(J(E_0, \beta_h^*) - \lambda I) = 0$ if and only if $\lambda_1 = -\mu_h < 0, \lambda_2 = -\mu_v < 0,\ \lambda_3 = -m_2 < 0,$
$\lambda_4 = -m_4 < 0$ and
$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$ (5.11)

Where,

$c_0 = 1,$
$c_1 = m_1 + m_2 + m_5$
$c_2 = m_2m_5 + m_1(m_2 + m_5) - \frac{\phi^2\omega^2\beta_h^*\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v}$
$c_3 = m_1m_2m_5(1 - R_0^2)$ (5.12)

If we also substitute 1(one) for R_0 in to equation (5.11), then it will have a simple zero eigenvalue and the other eigenvalues have negative real parts. Therefore; the disease-free equilibrium point E_0 is a non- hyperbolic.

To compute the coefficients equation (5.11) and equation (5.12), we determine the right and left eigenvectors corresponding to the zero eigenvalue. Thus, the components of the right eigenvectors denoted by w_i, for $i = 1, \dots, 7$ are given by

$$\begin{cases} -\mu_h w_1 + \varepsilon w_4 + \theta w_5 - m_{17}^* w_7 = 0 \\ -m_1 w_2 + m_{27}^* w_7 = 0 \\ -m_2 w_3 + m_{37}^* w_7 = 0 \\ \alpha w_2 + \sigma w_3 - m_3 w_4 = 0 \\ \gamma_s w_2 + \gamma_r w_3 - m_4 w_5 = 0 \\ -m_{62}(w_2 + w_3) - \mu_v w_6 = 0 \\ m_{72}(w_2 + w_3) - m_5 w_7 = 0 \end{cases} \quad (5.13)$$

Where, $m_{17}^* = \phi\omega\beta_h^*,\ m_{27}^* = \phi\omega\beta_h^*\rho(1-\pi), m_{37}^* = \phi\omega\beta_h^*(1-(1-\pi)\rho), m_{62}^* = m_{72} = \frac{\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$

$w_1 = \frac{\phi\omega\beta_h^*(\theta(\gamma_s m_2(1-\pi)\rho) - m_1m_2m_4)}{\mu_h m_1m_2m_4} w_7, w_2 = \frac{m_{27}^*}{m_1} w_7,\ w_3 = \frac{m_{37}^*}{m_2} w_7,\ w_4 = \frac{(m_1\sigma m_{37}^* + m_2\alpha m_{27}^*)}{m_1m_2m_3} w_7,$
$w_5 = \frac{\phi\omega\beta_h^*(\gamma_s m_2\rho(1-\pi) + \gamma_s(1-(1-\pi)\rho))}{m_1m_2m_4} w_7,\ w_6 = -\frac{m_5}{\mu_v} w_7,\ \ w_7 = w_7 > 0$ (5.14)

And the components of the left eigenvectors denoted by v_i, for $i = 1, \dots, 7$ are given by

$$\begin{cases} -\mu_h v_1 = 0 \\ -m_1 v_2 + \alpha v_4 + \gamma_s v_5 - m_{62}(v_6 - v_7) = 0 \\ -m_2 v_3 + \sigma v_4 + \gamma_s v_5 - m_{72}(v_6 - v_7) = 0 \\ \varepsilon v_1 - m_3 v_4 = 0 \\ \theta v_1 - m_4 v_5 = 0 \\ -\mu_v v_6 = 0 \\ -m_{17}^* v_1 + m_{27}^* v_2 + m_{37}^* v_3 - m_5 v_7 = 0 \end{cases} \quad (5.15)$$

$v_1 = v_4 = v_5 = v_6 = 0,$ for $v_7 = v_7 > 0, v_2 = \frac{m_{62}}{m_1} v_7,\ v_3 = \frac{m_{62}}{m_2} v_7$ (5.16)

Let we make the following change of state variables $S_h = x_1, I_{hs} = x_2, I_{hr} = x_3, T_h = x_4, R_h = x_5, S_v = x_6, I_v = x_7$ and using the vector notation $x = (x_1,\ x_2, x_3,\ x_4,\ x_5,\ x_6,\ x_7)^T$.The system (2.1) can then be written in the form $\frac{dx}{dt} = F(x)$ where, F = $(f_1,\ f_2, f_3,\ f_4,\ f_5,\ f_6,\ f_7)^T$ as shown below

$$\begin{cases} \frac{dx_1}{dt} = f_1 = \Lambda_h + \varepsilon x_4 + \theta x_5 - \left(\phi\omega\beta_h^* \frac{x_7}{x_1+x_2+x_3+x_4+x_5} + \mu_h\right) x_1 \\ \frac{dx_2}{dt} = f_2 = \rho(1-\pi)\phi\omega\beta_h^* \left(\frac{x_7}{x_1+x_2+x_3+x_4+x_5}\right) x_1 - (\mu_h + \delta_h + \alpha + \gamma_s) x_2 \\ \frac{dx_3}{dt} = f_3 = (1-(1-\pi)\rho)\phi\omega\beta_h^* \left(\frac{x_7}{x_1+x_2+x_3+x_4+x_5}\right) x_1 - (\mu_h + \delta_h + \gamma_r + \sigma) x_3 \\ \frac{dT_h}{dt} = f_4 = \alpha x_2 + \sigma x_3 - (\delta_h + \mu_h + \varepsilon) x_4 \\ \frac{dx_5}{dt} = f_5 = \gamma_s x_2 + \gamma_r x_3 - (\theta + \mu_h) x_5 \\ \frac{dx_6}{dt} = f_6 = \Lambda_v - \left(\phi\omega\beta_v \frac{x_2+x_3}{x_1+x_2+x_3+x_4+x_5} + \mu_v\right) x_6 \\ \frac{dx_7}{dt} = f_7 = \left(\phi\omega\beta_v \frac{x_2+x_3}{x_1+x_2+x_3+x_4+x_5}\right) x_6 - (\mu_v + \delta_v) x_7 \end{cases} \tag{5.17}$$

$$\frac{\partial^2 f_2}{\partial I_v \partial I_{hs}}(x_0,0) = \frac{\partial^2 f_2}{\partial I_v \partial I_{hr}}(x_0,0) = \frac{\partial^2 f_2}{\partial I_v \partial T_h}(x_0,0) = \frac{\partial^2 f_2}{\partial I_v \partial R_h}(x_0,0) = -\frac{\rho(1-\alpha\pi)\phi\omega\beta_h^*}{S_h^0}$$
$$\frac{\partial^2 f_3}{\partial I_v \partial I_{hs}}(x_0,0) = \frac{\partial^2 f_3}{\partial I_v \partial I_{hr}}(x_0,0) = \frac{\partial^2 f_3}{\partial I_v \partial T_h}(x_0,0) = \frac{\partial^2 f_3}{\partial I_v \partial R_h}(x_0,0) = -\frac{(1-(1-\alpha\pi)\rho)\phi\omega\beta_h^*}{S_h^0}$$
$$\frac{\partial^2 f_7}{\partial I_{hs} \partial S_h}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hs} \partial T_h}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hs} \partial R_h}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hr} \partial S_h}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hr} \partial T_h}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hr} \partial R_h}(x_0,0) =$$
$$-\frac{\phi\omega\beta_v S_v^0}{(S_h^0)^2}, \frac{\partial^2 f_7}{\partial I_h^2}(x_0,0) = \frac{\partial^2 f_7}{\partial R_h^2}(x_0,0) = -\frac{2\phi\omega\beta_v S_v^0}{(S_h^0)^2}, \frac{\partial^2 f_7}{\partial I_{hs} \partial S_v}(x_0,0) = \frac{\partial^2 f_7}{\partial I_{hr} \partial S_v}(x_0,0) = \frac{\phi\omega\beta_v}{S_h^0} \tag{5.18}$$

$$\frac{\partial^2 f_2}{\partial I_v \partial \beta_h^*}(x_0,0) = \phi\omega\rho(1-\alpha\pi),\ \frac{\partial^2 f_3}{\partial I_v \partial \beta_h^*}(x_0,0) = \phi\omega(1-(1-\alpha\pi)\rho) \tag{5.19}$$

$$a = v_2 w_7 \left(w_2 \frac{\partial^2 f_2}{\partial I_v \partial I_h}(x_0,0) + w_3 \frac{\partial^2 f_2}{\partial I_v \partial I_h}(x_0,0) + w_4 \frac{\partial^2 f_2}{\partial I_v \partial T_h}(x_0,0) + w_5 \frac{\partial^2 f_2}{\partial I_v \partial R_h}(x_0,0)\right) +$$
$$v_2 w_7 \left(w_2 \frac{\partial^2 f_3}{\partial I_v \partial I_h}(x_0,0) + w_3 \frac{\partial^2 f_3}{\partial I_v \partial I_h}(x_0,0) + w_4 \frac{\partial^2 f_3}{\partial I_v \partial T_h}(x_0,0) + w_5 \frac{\partial^2 f_3}{\partial I_v \partial R_h}(x_0,0)\right) +$$
$$w_1 v_7 \left(w_2 \frac{\partial^2 f_7}{\partial I_{hs} \partial S_h}(x_0,0) + w_3 \frac{\partial^2 f_7}{\partial I_{hr} \partial S_h}(x_0,0)\right) +$$
$$w_4 v_7 \left(w_2 \frac{\partial^2 f_7}{\partial I_{hs} \partial T_h}(x_0,0) + w_4 \frac{\partial^2 f_7}{\partial I_{hr} \partial T_h}(x_0,0)\right) + w_5 v_7 \left(w_2 \frac{\partial^2 f_7}{\partial I_{hs} \partial R_h}(x_0,0) +$$
$$w4\partial 2f7\partial Ihr\partial Rhx0,0v7w6w2\partial 2f7\partial Ihs\partial Svx0,0+v7w6w3\partial 2f7\partial Ihr\partial Svx0,0+v7w22\partial 2f6\partial Ihs2x0,0+v7w32\partial 2f6\partial Ihr2x0,0 \tag{5.20}$$

$$b = v_2 w_7 \frac{\partial^2 f_2}{\partial I_v \partial \beta_h^*}(x_0,0) + v_3 w_7 \frac{\partial^2 f_3}{\partial I_v \partial \beta_h^*}(x_0,0) \tag{5.21}$$

After substituting equation (5.14), equation (5.16) and equation (5.18) respectively in to equation (5.20), then the simplified values of the coefficient a in terms of w_7 and v_7 is given by

$$a = \frac{\mu_h \phi^2 \omega^2 \beta_h^* \beta_v}{\Lambda_h(\mu_h+\delta_h+\gamma_s)(\mu_h+\delta_h+\gamma_r)} v_7 w_7^2 \mathrm{B}_0$$

Where,

$$\mathrm{B}_0 = \frac{\Lambda_v(m_2\rho(1-\alpha\pi)+m_1(1-(1-\alpha\pi)\rho))}{\mu_v\mu_h m_3} - \frac{\phi\omega\beta_v\mu_h\Lambda_v(\rho(1-\pi)+(1-(1-\pi)\rho))((1-(1-\pi)\rho)m_1m_3(m_4+\gamma_r)+m_2\rho(1-\pi)(m_3+\gamma_s))}{m_1m_2\Lambda_h\mu_v}$$
$$-\frac{\phi\omega\beta_v\mu_h\Lambda_v(\rho^2(1-\alpha\pi)^2m_2^2+(1-(1-\pi)\rho)^2m_1^2)}{m_1m_2\Lambda_h\mu_v} - \frac{m_5}{\mu_v} - \frac{\phi\omega\beta_v\mu_h\Lambda_v(m_3(m_2\rho(1-\pi)\gamma_s+m_1(1-(1-\pi)\rho)\gamma_r)+m_2m_4\varepsilon\rho(1-\pi))}{\Lambda_h\mu_v m_1m_2m_3m_4}$$
$$-\frac{\theta\phi\omega\beta_v\Lambda_v(\rho(1-\pi)+(1-(1-\pi)\rho))(m_2\rho(1-\pi)\gamma_s+m_1(1-(1-\pi)\rho)\gamma_r)}{\Lambda_h\mu_v m_1m_2m_3m_4} \text{ and}$$

After substituting equation (5.14), equation (5.16) and equation (5.19) respectively in to equation (5.21), then the simplified values of the coefficient b in terms of w_7 and v_7 is given by

$$b = \frac{v_7^2 w_7 \phi^2 \omega^2 \beta_v \mu_h \Lambda_v(m_2\rho(1-\pi)+m_1(1-(1-\pi)\rho))}{\Lambda_h\mu_v m_1 m_2}$$

Clearly, the coefficient b is positive since all the parameters are non-negative. Thus, the local dynamics of the system (1) around E_0, for $\beta_h = \beta_h^*$ is depends on the sign of the coefficient a. Similar to theorem [24] we also established the following theorem.

Theorem 5 The system (1) will undergo backward bifurcation at $R_0 = 1$ if the coefficient a is positive ($B_0 > 0$) or ($R_c < R_0$) otherwise it will exhibit a forward bifurcation if a is negative ($B_0 < 0$).

5.2 Local stability of endemic equilibrium

Theorem 3: The endemic equilibrium point (E^*) of the system (2.1) is locally asymptotically stable if $R_0 > 1$.

Proof: The Jacobian matrix evaluated as

$$J(E^*) = \begin{pmatrix} -P & 0 & 0 & \varepsilon & \theta & 0 & -Q \\ R & -m_1 & 0 & 0 & 0 & 0 & S \\ T & 0 & -m_2 & 0 & 0 & 0 & Z \\ 0 & \alpha & \sigma & -m_3 & 0 & 0 & 0 \\ 0 & \gamma_s & \gamma_r & 0 & -m_4 & 0 & 0 \\ 0 & -X & -X & 0 & 0 & -Y & 0 \\ 0 & X & X & 0 & 0 & 0 & -m_5 \end{pmatrix} \tag{5.22}$$

$$P = \omega\phi\beta_h \frac{I_v^*}{N_h^*} + \mu_h,\ R = \rho(1-\pi)\omega\phi\beta_h \frac{I_v^*}{N_h^*},\ T = (1-(1-\pi)\rho)\omega\phi\beta_h \frac{I_v^*}{N_h^*},\ Q = \omega\phi\beta_h \frac{S_h^*}{N_h^*},$$
$$S = \rho(1-\pi)\omega\phi\beta_h \frac{S_h^*}{N_h^*},\ Z = (1-(1-\pi)\rho)\omega\phi\beta_h \frac{S_h^*}{N_h^*},\ X = \omega\phi\beta_v \frac{S_v^*}{N_h^*}\left(1 - \frac{(I_{hs}^* + I_{hr}^*)}{N_h^*}\right),$$
$$Y = \omega\phi\beta_v \frac{(I_{hs}^* + I_{hr}^*)}{N_h^*} + \mu_v$$

The eigenvalues of the $J(E^*)$ are given by:

Clearly $\lambda_1 = -\left(\omega\phi\beta_v \frac{(I_{hs}^* + I_{hr}^*)}{N_h^*} + \mu_v\right) < 0$ and its associate characteristics equation is

$$\lambda^6 + e_1\lambda^3 + e_2\lambda^2 + e_3\lambda^3 + e_4\lambda^2 + e\lambda + e_6 = 0 \tag{5.23}$$

Where.

$$e_1 = P + m_1 + m_2 + m_3 + m_4 + m_5$$
$$e_2 = m_3m_4 + (m_3 + m_4)(P + m_1 + m_2 + m_5) + (P + m_1)(m_2 + m_5) + m_2m_5 - XU$$
$$e_3 = m_3m_4 + (m_2 + m_5)(m_3m_4 + (P + m_1)(m_3 + m_4) + Pm_1) + Pm_1(m_3 + m_4) + (m_2m_5 - XZ)(P + m_1 + m_3 + m_4)$$
$$e_4 = m_3m_4(Pm_1 + m_2 + m_5 + m_2m_5 - XU) + (m_3 + m_4)\big(Pm_1 + (m_2 + m_5)Pm_1 + Pm_1(m_2m_5 - XZ)\big) + Pm_1(m_2m_5 - XZ) + \theta\gamma_sR(1 + m_2) + \theta\gamma_sT(1 + m_1) + (m_3 + m_5)\theta(\gamma_sR + \gamma_sT) + \varepsilon T\sigma(m_1 + m_4 + m_5) + Q(\gamma_sTm_1 + \gamma_rRm_2) + Q(m_3 + m_5)(\gamma_sT + \gamma_rR)$$
$$e_5 = (m_2 + m_5)Pm_1m_3m_4 + m_3m_4(m_2m_5 - XZ) + (m_3 + m_4)Pm_1(m_2m_5 - XZ) + \varepsilon T\sigma\big(m_1m_5 + m_4(m_1 + m_5)\big) + \varepsilon X(\sigma - \alpha)(RZ - ST) + \theta m_35(\gamma_rT + \gamma_sR) + \theta(m_3 + m_5)(\gamma_rTm_1 - \gamma_sRm_2) + Q(\gamma_sTm_1 + \gamma_rRm_2) + \theta X(\gamma_s - \gamma_r)(RZ - ST) + Q\left(RZX\big((\gamma_s - \gamma_r)\big) + (\gamma_rT - \gamma_sR)\right) m_2m_5$$
$$e_6 = Pm_1\, m_2m_3m_4m_5 + \varepsilon m_4\big(T\sigma + m_1m_5 + X(RZ - ST)\big)(\sigma - \alpha) + \theta m_3m_5(\gamma_rT + \gamma_sR) + (m_3 + m_5)\big(\theta(\gamma_sTm_1 + \gamma_rRm_2) + Q(\gamma_sTm_1 - \gamma_rRm_2)\big) + X(\gamma_s - \gamma_r)(\theta(RZM_5 - STm_3) + QRZm_5)$$

The Routh-Hurwitz criteria for Polynomial equation (5.23) will give six negative eigenvalues if the conditions given below are satisfied: $e_i > 0$, for i = 1, 2, 3, . . ., 6.The relevant Routh Hurwitz criteria in [25] could be used to show that the model system (2.1) is stable locally asymptotically when $R_0 > 1$.

6 Global Stability

6.1 Global stability of disease-free equilibrium point

To investigate the global stability of the disease-free equilibrium point E_0,we consider the Lyapunov function [26]. So that

$$V = m_5(m_2 I_{hs} + m_1 I_{hr})I_h + \phi\omega\beta_h(\rho(1-\pi)m_2 + (1-(1-\pi)\rho)m_1)I_v$$
$$\frac{dV}{dt} = m_5\left(m_2\frac{dI_{hs}}{dt} + m_1\frac{dI_{hs}}{dt}\right) + \phi\omega\beta_h\rho\left((1-\pi)m_2 + (1-(1-\pi)\rho)m_1\right)\frac{dI_v}{dt} \quad (6.1)$$

After substituting $\frac{dI_{hs}}{dt}$, $\frac{dI_{hr}}{dt}$ and $\frac{dI_v}{dt}$ from (1) to (35) and simplifying it, then we get

$$\frac{dV}{dt} = \frac{\phi^2\omega^2\beta_h\beta_v S_v^0\left(m_2\rho(1-\pi)+m_1(1-(1-\pi)\rho)\right)(I_{hs}+I_{hr})}{S_h^0} - m_1m_2m_5 \quad (6.2)$$

Since $\frac{dS_h}{dt} \leq \frac{(1-\varphi)\Lambda_h}{\mu_h} = S_h^0 = N_h^0$, $\frac{dS_v}{dt} \leq \frac{\Lambda_v}{\mu_v} = S_v^0$ (36) is equivalent to

$$\frac{dV}{dt} = \frac{\phi^2\omega^2\beta_h\beta_v\Lambda_v\mu_h\left(m_2\rho(1-\pi)+m_1(1-(1-\pi)\rho)\right)(I_{hs}+I_{hr})}{\Lambda_h\mu_h} - m_1m_2m_5 \quad (6.3)$$

Since $R_0^2 = \frac{\phi^2\omega^2\beta_h\beta_v\Lambda_v\mu_h m_2\rho(1-\pi)+m_1(1-(1-\pi)\rho)}{\Lambda_h\mu_h m_1m_2m_5}$, (37) is also equivalent to

$$\frac{dV}{dt} = m_1m_2m_5(R_0^2 - 1) \quad (6.4)$$

Therefore; $\frac{dV}{dt} \leq 0$ provided $(R_0^2 - 1) \leq 0$ which leads to $R_0 \leq 1$. $\frac{dV}{dt} = 0$ if and only if $I_{hs} = I_{hr} = 0$ or $R_0 = 1$.

Therefore; by Lasalle's invariant principle [27], every solution to equations of the model system(2.1) with initial conditions in Ω approaches to the disease-free equilibrium point E_0 at time t leads infinity whenever, $R_0 \leq 1$.Hence,the disease-free equilibrium E_0 is globally asymptotically stable if $R_0 \leq 1$.

Theorem 6 The disease-free equilibrium point E_0 of system (2.1) is globally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$.

The epidemiological implication of theorem 6 is that the elimination of the malaria disease is possible regardless of initial condition system (2.2) of the sub-population of the model system (2.1) whenever $R_0 \leq 1$.

7 Analysis of the Model with Optimal Control

In this section, we consider model system (2.1) and incorporate optimal combinations of time dependent control measures namely, (i) prevention measure for drug resistance $u_1(t) = u_1$ to minimize the proportion of the emergence of drug resistant of malaria strains as well as spread of the disease dynamics. This includes improving the way drugs used though improving prescribing, follow up practices and patient compliance, (ii) the use of insecticide treated bed net (ITN) $u_2(t) = u_2$ as preventive measure i.e., to reduce the number of bites from mosquitoes as they physically provide a barrier between the infectious mosquitoes and the susceptible humans, and also to reduce the population of the mosquitoes by killing them after they land on the treated net. (iii) treatment with drugs $u_3(t) = u_3$, treating individuals who developed symptoms of the disease, and (iv) the use of indoor residual spray (IRS), $u_4(t) = u_4$ as preventive measure i.e., insecticide spray on the breeding site of mosquitoes reduces the number of mosquito populations by killing these rest indoors after feeding. The controls are practiced on time interval $[t_0, t_f]$, where t_0 and t_f are initial and final time respectively. After incorporating the above controls in to the basic model (2.1) we get the following modified state equations:

$$\begin{cases} \frac{dS_h}{dt} = \Lambda_h + \big(\varepsilon + u_1 + (1-\tau u_3)\big)T_h + \theta R_h - ((1-u_2)\lambda_h + \mu_h)S_h \\ \frac{dI_{hs}}{dt} = (1-u_2)\rho(1-(1-u_1)\pi)\lambda_h S_h - \big(\mu_h + \delta_h + \alpha + (\gamma_s+\tau u_3)\big)I_{hs} \\ \frac{dI_{hr}}{dt} = (1-u_2)(1-(1-(1-u_1)\pi)\rho)\lambda_h S_h - (\mu_h + \delta_h + \sigma + \gamma_r)I_{hr} \\ \frac{dT_h}{dt} = \alpha I_{hs} + \sigma I_{hr} - \big(\delta_h + \mu_h + \varepsilon + u_1 + (1-\tau u_3)\big)T_h \\ \frac{dR_h}{dt} = (\gamma_s+\tau u_3)I_{hs} + \gamma_r I_{hr} - (\theta + \mu_h)R_h \\ \frac{dS_v}{dt} = \Lambda_v - \big((1-u_2)\lambda_v + \mu_v + \delta u_2 + \beta u_4\big)S_v \\ \frac{dI_v}{dt} = (1-u_2)\lambda_v S_v - (\mu_v + \delta_v + \delta u_2 + \beta u_4)I_v \end{cases} \tag{7.1}$$

Here the following objective function J is used to minimize the number of infected human with drug sensitive and drug resistance malaria parasite strains, infective in treatment human populations and total mosquito populations while keeping the costs of applying the controls u_1, u_2, u_3 and u_4 as low as possible.

$$J = min \int_0^{t_f} \left(A_1 I_{hs} + A_2 I_{hr} + A_3 T_h + A_4 I_v + \tfrac{1}{2}\textstyle\sum_1^4 d_i u_i^2\right) dt \tag{7.2}$$

Where, i =1,2,3,4, A_1, A_2, A_3 and A_4 and d_1, d_2, d_3 and, d_4 are coefficients associated to the state variable and controls respectively. Following the approach [28,29], the cost of the controls have been chosen quadratic.

Thus,the goal is to find, an optimal control quadruple, u_1^*, u_2^*, u_3^* and u_4^* such that

$$J(u_1^*, u_2^*, u_3^*, u_4^*) = min\{J(u_1, u_2, u_3, u_4): u_1, u_2, u_3, u_4 \epsilon U\} \tag{7.3}$$

Where, $U = \{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t) \quad : 0 \le u_i < 1, i = 1,2,\dots,4, 0 \le t \le t_f\}$ is the control set.

The Pontryagins's Maximum Principle [29] converts the system (7.1) with equation (7.2) and equation (7.3) into a problem of minimizing pointwise the Hamiltonian H with respect to u_1, u_2, u_3 and u_4

$$H = (S_h, I_{hs}, I_{hr}, T_h, R_h, S_v\, I_v, t) = L(I_{hs}, I_{hr}, T_h, I_v, u_1, u_2, u_3, u_4, t) + \lambda_1 \frac{dS_h}{dt} + \lambda_2 \frac{dI_{hs}}{dt} + \lambda_3 \frac{dI_{hr}}{dt} + \lambda_4 \frac{dT_h}{dt} + \lambda_5 \frac{dR_h}{dt} + \lambda_6 \frac{dS_v}{dt} + \lambda_7 \frac{dI_v}{dt} \tag{7.4}$$

Where, $L(I_{hs}, I_{hr}, T_h, I_v, u_1, u_2, u_3, u_4, t) = A_1 I_{hs} + A_2 I_{hr} + A_3 T_h + A_4 I_v + \frac{1}{2}\sum_1^4 d_i u_i^2$ for i=1,2,3,4

and λ_i, for i= 1,2,3,4,5,6,7 are adjoint variable. Using the exitance result for the optimal control [29], we established the following theorem as

Thereom7 There exists a set of an optimal control $u_i^* = (u_1^*, u_2^*, u_3^* \quad u_4^*)$ and corresponding state solution, S_h^*, I_{hs}^*, I_{hr}^*, T_h^* R_h^* S_v^* and I_v^* that minimizes $J(u_1, u_2, u_3, \quad u_4)$ over U subject to system (7.1). Further, there exists adjoint functions $\lambda_1(t), \dots, \lambda_7(t)$, and $u_1(t), \dots u_4(t)$ satisfying

$$\begin{cases} \frac{d\lambda_1}{dt} = \mu_h\lambda_1 + (1-u_2)\lambda_h(\lambda_1 - \rho(1-(1-u_1)\pi)\lambda_2 - (1-(1-(1-u_1)\pi)\rho)\lambda_3) \\ \frac{d\lambda_2}{dt} = -A_1 + \alpha(\lambda_2 - \lambda_4) - (\gamma_s + \tau u_3)(\lambda_2 - \lambda_5) + (\mu_h + \delta_h)\lambda_2 + \frac{(1-u_2)S_v}{N_h}\left(1 - \frac{I_{hs}}{N_h}\right)(\lambda_6 - \lambda_7) \\ \frac{d\lambda_3}{dt} = -A_2 + \sigma(\lambda_3 - \lambda_4) - \gamma_r(\lambda_3 - \lambda_5) + (\mu_h + \delta_h)\lambda_2 + \frac{(1-u_2)S_v}{N_h}\left(1 - \frac{I_{hr}}{N_h}\right)(\lambda_6 - \lambda_7) \\ \frac{d\lambda_4}{dt} = -A_3 + \big(\varepsilon + u_1 + (1-\tau u_3)\big)(\lambda_4 - \lambda_1) + (\mu_h + \delta_h)\lambda_4 \\ \frac{d\lambda_5}{dt} = \mu_h\lambda_5 + \theta(\lambda_5 - \lambda_1) \\ \frac{d\lambda_6}{dt} = (1-u_2)\lambda_v(\lambda_6 - \lambda_7) + (\mu_v + \delta u_2 + \beta u_4)\lambda_6 \\ \frac{d\lambda_7}{dt} = (1-u_2)\frac{\phi\omega\beta_h S_h}{N_h}(\lambda_1 - \rho(1-(1-u_1)\pi)\lambda_2 - (1-(1-(1-u_1)\pi)\rho)\lambda_3) \\ \qquad + (\mu_v + \delta_v + \delta u_2 + \beta u_4)\lambda_7 - A_4 \end{cases} \tag{7.5}$$

with transversality conditions

$$\lambda_i(t_f) = 0 \text{ for } i = 1,2,3,4,5,6,7 \tag{7.6}$$

Further, the optimal controls u_1^*, u_2^*, u_3^* and u_4^* are given by

$$\begin{aligned} u_1^* &= min\left\{max\left(0, \frac{(1-u_2)\pi\rho\lambda_h S_h(\lambda_3-\lambda_2) + (\lambda_4-\lambda_1)T_h}{d_1}\right), 1\right\} \\ u_2^* &= min\left\{max\left(0, \frac{\lambda_h S_h\left((1-(1-u_1)\pi)\rho\lambda_2+(1-(1-(1-u_1)\pi)\rho)\lambda_3-\lambda_1\right)+\lambda_v S_v(\lambda_7-\lambda_6)+\delta(S_v\lambda_6+I_v\lambda_7)}{d_2}\right), 1\right\} \\ u_3^* &= min\left\{max\left(0, \frac{\tau(\lambda_1-\lambda_4)T_h+\tau(\lambda_2-\lambda_5)I_{hs}}{d_3}\right), 1\right\} \\ u_4^* &= min\left\{max\left(0, \frac{\beta(S_v\lambda_6+I_v\lambda_7)}{d_4}\right), 1\right\} \end{aligned} \tag{7.7}$$

Proof:

The existence of the optimal control follows from Fleming and Rischel [30] due to convexity of the integrand objective functional J in (7.2) with respect to u_i, $i = 1,2,3,4$ over the convex and closed control set U and the system (7.1) satisfies the and Lipchitz property with respect to state variables since the state solutions are bounded. The differential equation (7.5) governing the adjoint variables $\lambda_1, \lambda_2, \ldots, \lambda_7$ are obtained by partial differentiation of the Hamiltonian H equation(7.4) with respect to the corresponding state variables that is,

$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S_h}$, $\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I_{hs}}$, $\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_{hr}}$, $\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial T_h}$, $\frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R_h}$, $\frac{d6}{dt} = -\frac{\partial H}{\partial S_v}$, $\frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial I_v}$ with terminal conditions eqation(7.6).The characterization of optimal control given by system (7.7) is obtained by partial derivative of the Hamiltonian H equation (7.4) with respect to each control u_i and solving $\frac{\partial H}{\partial u_i} = 0$, for $i = 1,2,3,4$.

8 Numerical Simulation

In this section, numerical simulations are performed to confirm with our analytical results stated in the optimality system which is characherized by the state system(7.1) and the adjoint system (7.5) was solved numerically by applying Runge Kutta fourth order schemes of the approach [31]. The implimentation of the scheme was done using MATLAB packege.

The parameters values provided in Table 1 are used so that $R_0 = 1.1937255489 > 1$. The simulations of the model are done by using the initial conditions given by $S_h(0) = 800$, $I_{hs}(0) = 30$, $I_{hr}(0) = 30$, $T_h(0) = 30$, $R_h(0) = 10$, $S_v(0) = 5000$, $I_v(0) = 100$. To minimize malaria infectious humans and the total mosquito populations as well as minimizing the associated costs of controls, the weights constant values in the objective function(38) are chosen so that $A_1 = A_2 = A_3 = A_4 = d_1 = d_2 = d_3 = d_4 = 4$.

In order to analyze the numerical results, we proposed optimal combinations of the aforementioned control strategies as alternative choose to minimize the spread of malaria disease dynamics. So as to do this, we introduced different optimal combination strategies in our model and numerically compare their effects on malaria infeected populatons.Thus,the proposed optimal combinations and numerical result analysis are as follows

- **Strategy *a*:** Combination of use of preventive control of drug resistance, insecticide treated net ITN and treatment of infective individuals
- **Strategy *b*:** Combination of use of preventive control of drug resistance, indoor residual spray IRS for vector control and treatment of infective individuals
- **Strategy *c*:** Combination of use of insecticide treated nets ITN, indoor residual spray IRS for vector control and treatment of infective individuals
- **Strategy *d*:** Combination of use of preventive control of drug resistance, insecticide treated nets ITN and indoor residual spray IRS and treatment of infective individuals

Table 1. Lists of parameters of the model system (2.1)

Parameter symbol	Value	Source
β_h	0.8333	[32]
μ_h	0.00005447	[33]
δ_h	0.0680	[34]
γ_s	0.0022	[35]
γ_r	0.00019	[36]
τ	0.5000	Assumed
ω	0.2000	[37]
ϕ	0.5020	[37]
Λ_v	0.0710	[38]
δ_v	0.0100	[39]
μ_v	0.0500	[40]
Λ_h	0.00000575	[33]
θ	0.01672	[41]
ρ	0.7000	[42]
β_v	0.48	[41]
β	0.2500	Assumed
δ	0.2500	Assumed
α	0.0500	Assumed
ε	0.0500	Assumed
π	0.5000	Assumed
σ	0.0500	Assumed

8.1 Strategy a

Control with the preventive of drug resistance, insecticide treated net ITN, and treatment($u_1 \neq 0,\ u_2 \neq 0, u_3 \neq 0, u_4 = 0$). In this strategy, we compare the strategy a situation where no control ($u_1 = 0,\ u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy a . It can be seen from the Figs. 3a, 3b, 3c, 3d, 3e and 3f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. From this, one can observe that strict application of strategy a for a period between 10 and 30 days is sufficient to reduce the number of individuals with malaria symptoms and malaria infected vectors to zero. It can be noted that, a combination of preventive of drug resistance, insecticide treated nets ITN, and treatment can play an important role in minimizing malaria infectious. The control profile shown in Fig. 3g shows that, controls $u_1,\ u_2$ and u_3 decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance u_1, insecticide treated net ITN u_2 and medical treatment u_3 of individuals under this strategy.

8.2 Strategy b

Control with the preventive of drug resistance, indoor residual spray IRS, and treatment($u_1 \neq 0,\ u_2 = 0, u_3 \neq 0, u_4 \neq 0$). In this strategy, we compare the strategy a situation where no control ($u_1 = 0,\ u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy b. It can be seen from the Figs. 4a,4b,4c,4d ,4e and 4f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. Even though this strategy minimizes the number of malaria infectious populations, however, it is not enough to eliminate the disease at a given time and hence there is a need for additional control effort to eliminate the disease out of the community. The control profile shown in Fig. 4g shows that, controls $u_1,\ u_3$ and u_4 decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance u_1, indoor residual spray IRS u_4 and medical treatment u_3 of individuals under this strategy.

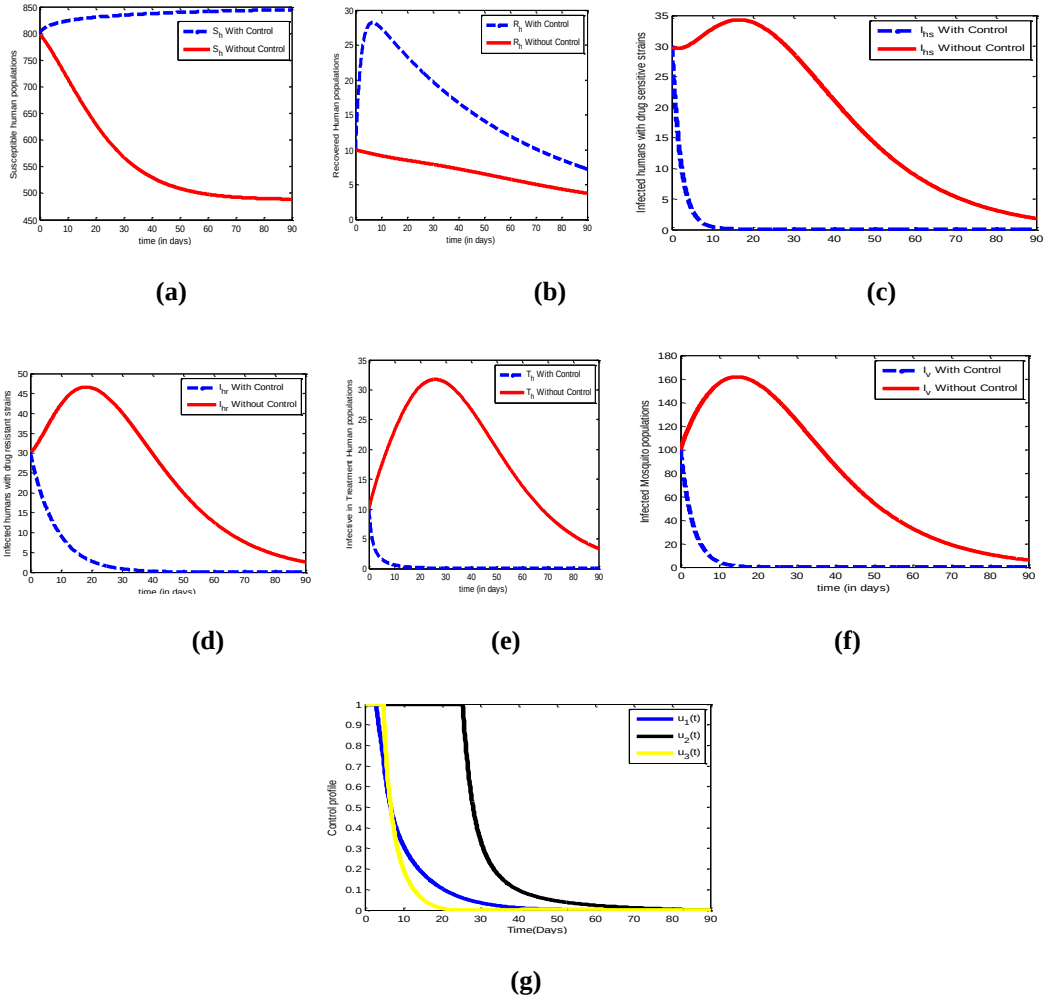


Fig. 3. Simulations of the model Showing the effect of preventive control of drug resistance, insecticide treated net ITN and Treatment controls

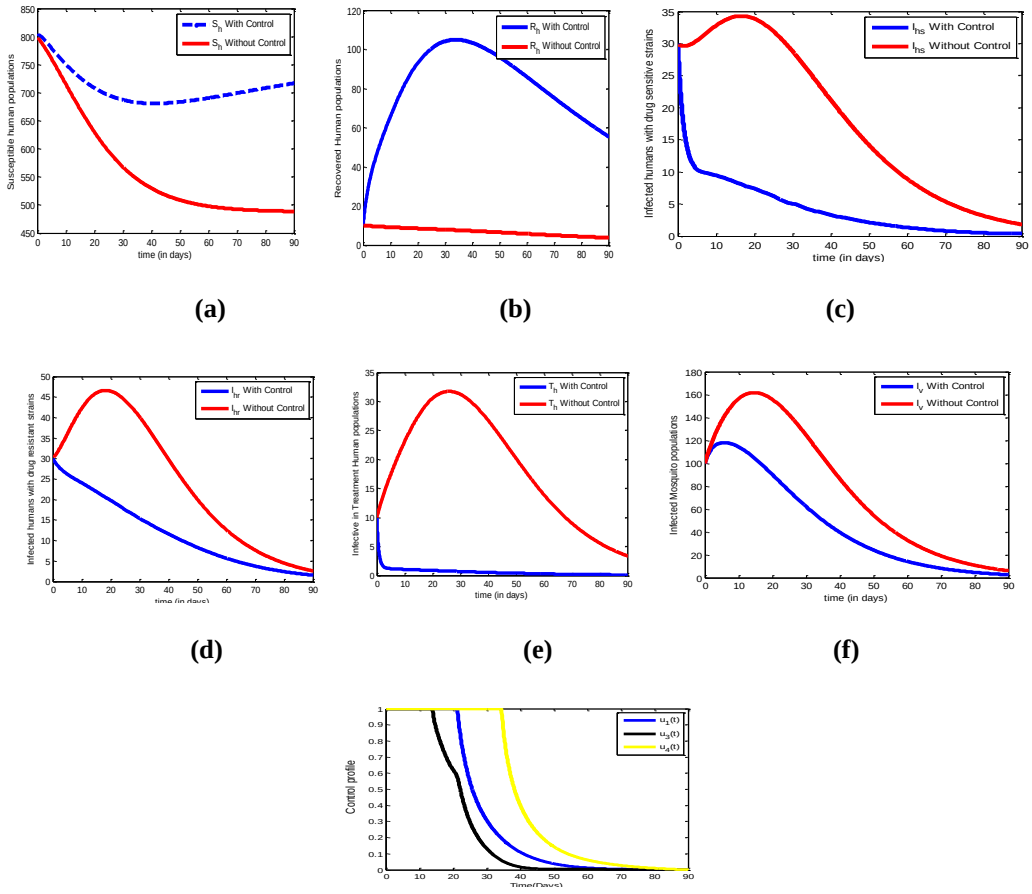
8.3 Strategy *c*

Control with insecticide treated net ITN, indoor residual spray IRS, and treatment($u_1 = 0,\ u_3 \neq 0, u_3 \neq 0, u_4 \neq 0$) In this strategy, we compare the strategy a situation where no control($u_1 = 0,\ u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy c . It can be seen from the Figs. 5a,5b,5c,5d ,5e and 5f that there is a significant increase in the number of susceptible and recovered human populations and a dramatic decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy c, I_{hs}, T_h and I_v within time $t = 10$ days, I_{hr} within time $t = 30$ days will be eliminated from the system. This result is a bit more promising than strategy a and strategy b. The control profile shown in Fig. 5g shows that, control u_3 is at 50% initially and decreases from the maximum of 70% to the lower bound while controls u_2 and u_4 decreases from the maximum of 100% to the lower bound within 90 days. This suggests that, a high effort is required for the use of insecticide treated net u_2, and indoor residual spray IRS u_4 for vector control and there is a low effort for the use of medical treatment u_3 of individuals under this strategy.

8.4 Strategy *d*

Control with the preventive of drug resistance, insecticide treated net ITN, indoor residual spray IRS, and treatment ($u_1 \neq 0,\ u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$) . In this strategy, we compare the strategy a situation where no

control ($u_1 = 0,\ u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy d. It can be seen from the Figs, 6a,6b,6c,6d ,6e and 6f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy d, I_v, I_{hs}, T_h and I_{hr} within time $t = 8$,10 ,11 and 30 days respectively will be eliminated from the system. This result is a bit more promising than when strategy a and strategy b except possibly strategy c which yield almost the same results. The control profile shown in Fig. 6g shows that, controls u_1, u_2 u_3 and u_4 decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance u_1, insecticide treated net u_2 ,indoor residual spray IRS u_4 and medical treatment u_3 of individuals under this strategy.

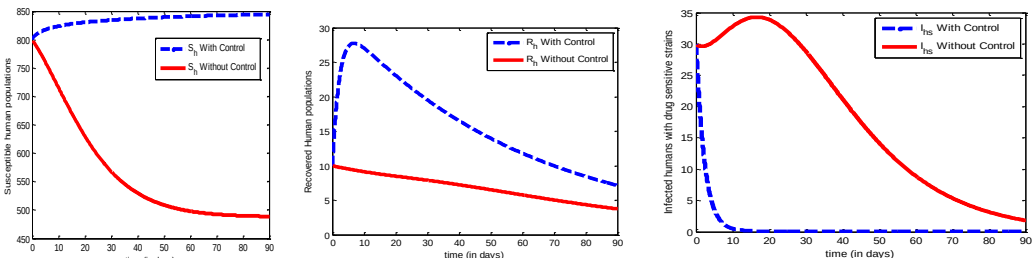


(a) **(b)** **(c)**

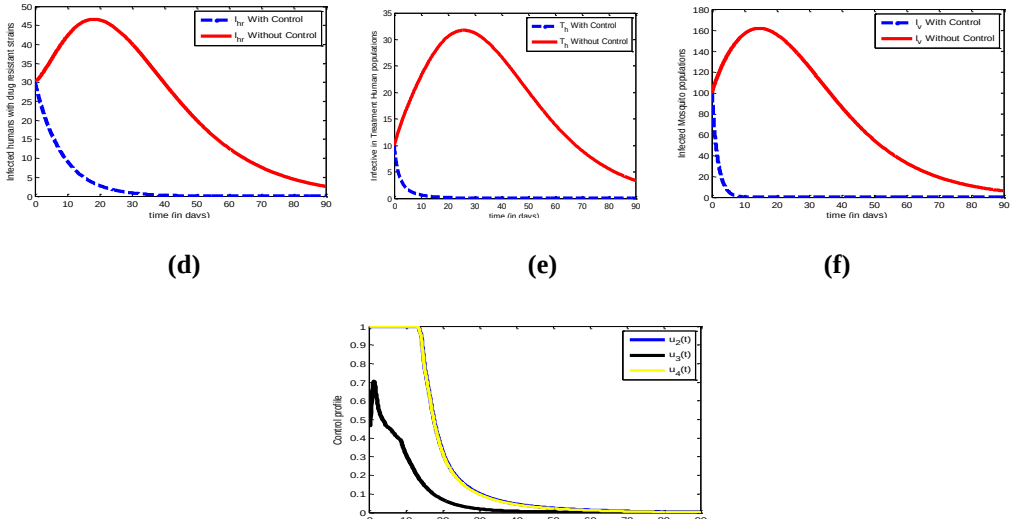
(d) **(e)** **(f)**

(g)

Fig. 4. Simulations of the model Showing the effect of preventive control of drug resistance, indoor residual spray IRS and treatment controls



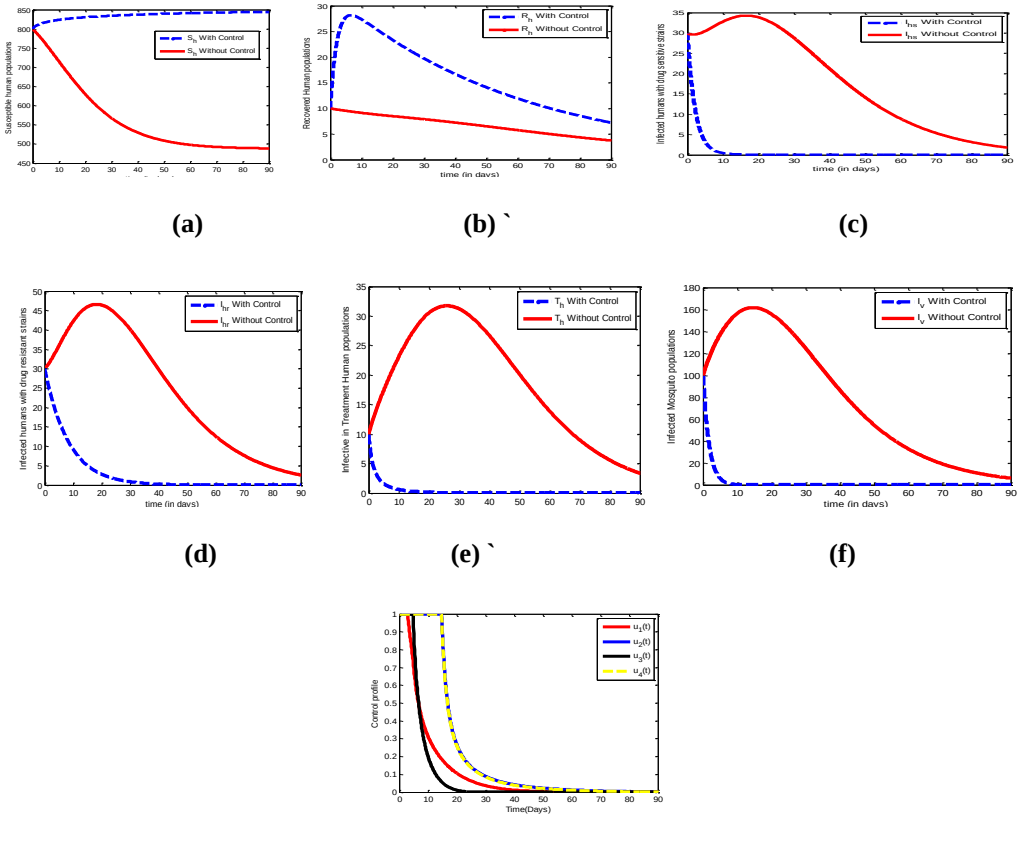
(a) **(b)** **(c)**



(d) **(e)** **(f)**

(g)

Fig. 5. Simulations of the model showing the effect of insecticide treated net, indoor residual spray IRS and treatment controls



(a) **(b) `** **(c)**

(d) **(e) `** **(f)**

(g)

Fig. 6. Simulations of the model Showing the effect of preventive control of drug resistance, insecticide treated net, indoor residual spray and treatment controls

9 Discussions and Conclusions

In this study, a non-linear system of ordinary differential equation model that describes the dynamics of malaria disease transmission is formulated and analyzed. Conditions are derived from the existence of disease-free and endemic equilibria. The basic reproduction number R_0 of the model is obtained, and we investigated that it is a threshold parameter between the extinction and persistence of the disease. If R_0 is less than unity, then the disease-free equilibrium point is both locally and globally asymptotically stable resulting in the disease removing out of the host populations. The disease can persist whenever R_0 is greater than unity. Furthermore, at R_0 is equal to unity, existence conditions are derived from the endemic equilibrium for both forward and backward bifurcations.

The numerical simulations of the optimality system which is characherized by the state system (7.1) and the adjoint system (7.5) was solved numerically by applying Runge Kutta fourth order schemes ,The result of numerical simulations of these can be seen from the Fig. 6 that the combination of prevention of drug resistance, insecticide treated net ITN, indoor residual spray IRS and active treatment or strategy **d** performs the best to control the disease in given time period of intervention. Finally, we note that with the strict application of either one of the incorporated combinations of optimal control strategies, it is possible to reduce the number populations with malaria symptoms to zero in the given time and the spread of the disease dynamics. Further we note that, application of optimal control strategy is not only reduce the number populations with malaria symptoms but also it reduces the emergence of drug resistant malaria strains as well as the spread of the disease.

10 Recommendations

Here we recommend to malaria control policy makers, health care workers and any concerning body may use the incorporated strategy in this paper to dwindle the malaria disease burden on the community.

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Completing Interests

Authors have declared that no completing interests exist.

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