Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 5560543, 9 pages
https://doi.org/10.1155/2021/5560543

Research Article

Beta Operator with Caputo Marichev-Saigo-Maeda Fractional Differential Operator of Extended Mittag-Leffler Function

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Received 10 January 2021; Revised 20 February 2021; Accepted 26 February 2021; Published 13 March 2021

Academic Editor: Soheil Salahshour

In this paper, a beta operator is used with Caputo Marichev-Saigo-Maeda (MSM) fractional differentiation of extended Mittag-Leffler function in terms of beta function. Further in this paper, some corollaries and consequences are shown that are the special cases of our main findings. We apply the beta operator on the right-sided MSM fractional differential operator and on the left-sided MSM fractional differential operator. We also apply beta operator on the right-sided MSM fractional differential operator with Mittag-Leffler function and the left-sided MSM fractional differential operator with Mittag-Leffler function.

1. Introduction

Fractional calculus is a fast-growing field of mathematics that shows the relations of fractional-order derivatives and integrals. Fractional calculus is an effective subject to study many complex real-world systems. In recent years, many researchers have calculated the properties, applications, and extensions of fractional integral and differential operators involving the various special functions.

Integral and differential operators in fractional calculus have become a research subject in recent decades due to the ability to have arbitrary order. Special functions are the functions that have improper integrals or series. Some of the well-known functions are the gamma function, beta function, and hypergeometric function.

Many researchers establish compositions of new fractional derivative formula called MSM Caputo-type fractional operators on well-known functions like the Mittag-Leffler function.

Owolabi [1] studied the dynamic evolution of chaotic and oscillatory waves arising from dissipative dynamical systems of elliptic and parabolic types of partial differential equations. Owolabi [2] deals with the numerical solution of space-time-fractional reaction-diffusion problems used to model some complex phenomena that are governed by the dynamic of anomalous diffusion. The time- and space-fractional reaction-diffusion equation is modeled by replacing the first-order derivative in time and the second-order derivative in space, respectively. Owolabi and Shikongo [3] expanded the studies on a tumor-host model with chemotherapy application, by considering a model which includes terms that can express both intrinsic drug resistance and drug-induced resistance. Owolabi et al. [4] studied chaotic dynamical systems. In the models, integer-order time derivatives are replaced with the Caputo fractional-order counterparts. A Chebyshev spectral method is presented for the numerical approximation. Owolabi [5] is concerned with the formulation and analysis of a reliable numerical method based on the novel alternating direction implicit finite difference scheme for the solution of the fractional reaction-diffusion system. The integer first-order derivative in time is replaced with the Caputo fractional derivative operator.

Yavuz [6] investigated the novel solutions of fractional-order option pricing models and their fundamental mathematical analyses. The main novelties of this paper are the analysis of the existence and uniqueness of European-type option pricing models providing to give fundamental solutions to them and a discussion of the related analyses by considering both the classical and generalized Mittag-Leffler kernels. Yavuz and Abdeljawad [7] presented a fundamental solution method for nonlinear fractional regularized long-wave (RLW) models. Since analytical methods cannot be applied easily to solve such models, numerical or semianalytical methods have been extensively considered in the literature.

Sene and Srivastava [8] presented a new stability notion of the fractional differential equations with exogenous input. Motivated by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, Sene [9] addresses new applications of the generalized Mittag-Leffler input stability to the fractional-order electrical circuits. He considered the fractional-order electrical circuits in the context of the generalized Caputo-Liouville derivative.

Singh [10] deals with certain new and interesting features of the fractional blood alcohol model associated with the powerful Hilfer fractional operator. The solution of the model depends on three parameters such as (i) the initial concentration of alcohol in the stomach after ingestion, (ii) the rate of alcohol absorption into the bloodstream, and (iii) the rate at which the alcohol is metabolized by the liver. Singh et al. [11] studied the solution of the local fractional Fokker-Planck equation (LFFPE) on the Cantor set. They performed a comparison between the reduced differential transform method (RDTM) and local fractional series expansion method (LFSEM) employed to the LFFPE. Singh et al. [12] analyzed the local fractional Poisson equation (LFPE) by employing the q-homotopy analysis transform method (q-HATM). They have studied PE in the local fractional operator sense. To handle the LFPE, some illustrative example was discussed.

Kumar et al. [13] deal with a fractional extension of the vibration equation for very large membranes with distinct special cases. A numerical algorithm based on the homotopic technique is employed to examine the fractional vibration equation. The stability analysis is conducted for the suggested scheme.

Kilbas et al. [14] have been working on the composition of Riemann-Liouville fractional integration and differential operators. Rao et al. [15] introduced the result that fractional integration and fractional differentiation are interchanged. Agarwal and Jain [16] developed fractional calculus formula of polynomial using the series expansion method. Further, it is expressed in terms of Hadamard product.

Nadir and Khan [17] applied Caputo-type MSM fractional differentiation on Mittag-Leffler function. Mondal and Nisar [18] applied the Marichev-Saigo-Maeda operator on the Bessel function. Nadir and Khan [19] applied the Marichev-Saigo-Maeda differential operator and generalized incomplete hypergeometric functions.

Nadir et al. [20] studied the extended versions of the generalized Mittag-Leffler function. Nadir and Khan [21, 22] used fractional integral operator associated with extended Mittag-Leffler function. Nadir and Khan [23] applied Weyl fractional calculus operators on the extended Mittag-Leffler function.

Srivastava et al. [24] defined the following function:

$$\Theta\left(\{k_n\}_{n\in N_0}\,;x\right) := \left\{\begin{array}{ll} \sum_{n=0}^{\infty} k_n \dfrac{z^n}{n!} & \begin{pmatrix} |x<\mathfrak{R}| \\ 0<\mathfrak{R}<\infty \\ k_0 := 1 \end{pmatrix} \\ m_0 x \varpi \exp(x)\left[1+0\left(\dfrac{1}{2}\right)\right] & \begin{pmatrix} \mathfrak{R}(x)\to\infty \\ m_0>0\,;\varpi\in\mathbb{C} \end{pmatrix} \end{array}\right\}, \tag{1}$$

where $\Theta(\{k_n\}_{n\in N_0}\,;x)$ is considered to be analytical with $|x| < \mathfrak{R}, 0<\mathfrak{R}<\infty \{k_n\}_{n\in N_0}$ which is a sequence of Taylor-Maclaurin coefficients m_0 and ϖ which are constants and depend upon the bounded sequence $\{k_n\}_{n\in N_0}$.

As the series,

$$E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(x,p) = \sum_{k=0}^{\infty} \frac{B_\rho^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma:\rho)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}, \tag{2}$$

where

$$\begin{pmatrix} x,\mu,\gamma\in C\,;\mathfrak{R}(\varepsilon)>0 \\ \mathfrak{R}(\mu)>0, \mathfrak{R}(\gamma)>1:\rho\geq 0 \end{pmatrix} \tag{3}$$

is known as the extension of the Mittag-Leffler function. It is defined by Parmar [25].

Mittag-Leffler function with special cases is given as follows.

(i) When $K_n=(\rho)_n/(\sigma)_n$, then the extended form of Equation (2) takes the form:

$$E_{\varepsilon,\mu}^{(\rho,\sigma):\,\gamma}(x\,;\rho) = \sum_{k=0}^{\infty} \frac{B_\rho^{(\rho,\sigma)}(\gamma+k,1-\gamma:\rho)}{B(\gamma,1-\gamma)}\,\frac{x^k}{\Gamma(\varepsilon k+\mu)}. \tag{4}$$

Under the condition,

$$\begin{pmatrix} x,\mu,\gamma\in C\,;\mathfrak{R}(\varepsilon)>0,\mathfrak{R}(\sigma)>0 \\ \mathfrak{R}(\varepsilon)>0,\mathfrak{R}(\mu)>0,\mathfrak{R}(\gamma)>1\,;\rho\geq 0 \end{pmatrix}. \tag{5}$$

(ii) If we select a bounded sequence $K_n=1$, then Equation (2) reduces to the definition of Özarslan and Yilmaz [26]

$$E_{\varepsilon,\mu}{}^{\gamma}(x;\rho)=\sum_{k=0}^{\infty}\frac{B(\gamma+k,1-\gamma:\rho)}{B(\gamma,1-\gamma)}\,\frac{x^k}{\Gamma(\varepsilon k+\mu)},\quad\left(\begin{array}{c}x,\mu,\gamma\in C;\Re(\varepsilon)>0\\ \Re(\mu)>0,\Re(\gamma)>1:\rho\geq 0\end{array}\right). \tag{6}$$

(iii) Another special case of Equation (2) is when $K_n=1$ and $\rho=0$, then Equation (2) reduces to the Prabhakar's function (Prabhakar [20]) of three parameters:

$$E_{\varepsilon,\mu}{}^{\gamma}(x;\rho)=\sum_{k=0}^{\infty}\frac{(\gamma_k)x^k}{\Gamma(\varepsilon k+\mu)k!},\quad(\varepsilon,\mu,\gamma\in c;\Re(\varepsilon)>0,\Re(\mu)>0). \tag{7}$$

(iv) If we set $\varepsilon=\mu=1$, then our expression for $E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}E_{\varepsilon,\mu}^{(\rho,\sigma):\,\gamma}$ and $E_{\varepsilon,\mu}^{\gamma}$ reduces to the extended confluent hypergeometric functions:

$$\begin{aligned}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(x;p)&=\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}(\gamma;1;x),\\ E_{\varepsilon,\mu}^{((p,q);\gamma)}(x;p)&=\Phi_p^{(p,q)}(\gamma;1;x),\\ E_{1,1}^{\gamma}(x;p)&=\Phi_P(\gamma;1;x).\end{aligned} \tag{8}$$

2. The Confluent Hyper Geometric Function

The confluent hypergeometric function by Rainville [27] is well defined as ${}_2F_1(a,b;c;x)$ which is represented by hypergeometric series.

$${}_2F_1(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}.\frac{x^m}{m!}. \tag{9}$$

3. The Hadamard Product of the Power Series

As indicated in Pohlen [28],

let

$$\begin{aligned}g(z)&=\sum_{m=0}^{\infty}x_m z^m,\\ h(z)&=\sum_{m=0}^{\infty}y_m z^m\end{aligned} \tag{10}$$

be the two power series, then the Hadamard product of power series is defined as follows:

$$(g*h)(z)=\sum_{m=0}^{\infty}x_my_mz^m=(h.g)(z);(|z|<R), \tag{11}$$

where

$$R=\lim_{m\to\infty}\left|\frac{x_my_m}{x_{m+1}y_{m+1}}\right|\left(\lim_{k\to\infty}\left|\frac{x_m}{x_{m+1}}\right|\right)\left(\lim_{m\to\infty}\left|\frac{y_m}{y_{m+1}}\right|\right)=R_g.R_h, \tag{12}$$

where R_g and R_h stand for radii of convergence of the above series $g(z)$ and $h(z)$, respectively. Therefore, in general, it is to be noted that if the one power series is an analytical function, then the series of Hadamard products are also the same as an analytical function.

4. Beta Function

The beta function by Saigo and Maeda [29] is defined as follows:

$$B(f(z),p,q)=\int_0^1 z^{p-1}(1-z)^{q-1}dz=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{13}$$

5. Appell Function

Appell function by Rainville [27] of first kind F_3 is basically a two-variable hypergeometric function defined as follows:

$$\begin{aligned}F_3\left(\omega,\omega',\upsilon,\upsilon',\eta;x,y\right)&=\sum_{m,n}^{\infty}\frac{(\omega)_m\left(\omega'\right)_n(\upsilon)_m\left(\upsilon'\right)_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\\ &=\sum_{m=0}^{\infty}\frac{(\omega)_m(\upsilon)_m}{(\eta)_m}\,{}_2F_1\left[\begin{array}{c}\omega',\upsilon';\\ \eta+m;\end{array};y\right]\frac{x^m}{m!}.\end{aligned} \tag{14}$$

6. The Left-Sided MSM Fractional Differential Operator

The left-sided Marichev-Saigo-Maeda fractional differential operator containing Appell F_3 function in their kernel by Saigo and Maeda [29] is defined as follows:

Let

$$\begin{aligned}\alpha,\alpha_1,\omega,\omega_1,\mu,\rho\in C,x>0\left(D_{0+}^{\alpha,\alpha_1,\omega,\omega_1,\mu}f\right)(x)&=\left(I_{0+}^{-\alpha,-\alpha_1,-\omega,-\omega_1,-\mu}f\right)(x)\\ &=\frac{d^n}{dx^n}\left(I_{0+}^{\alpha,-\alpha_1,-\omega,-\omega_1+n,+\mu+n}f\right)(x),\end{aligned} \tag{15}$$

where $\Re(\mu)>0$ and $n=[\Re(-\mu)+1]$.

7. The Right-Sided MSM Fractional Differential Operator

The right-sided Marichev-Saigo-Maeda fractional differential operator containing Appell function F_3 in their kernel by Saigo and Maeda [29] is as follows:

Let

$$\alpha, \alpha_1, \omega, \omega_1, \mu, \rho \in C, x > 0\left(D_{0-}^{\alpha,\alpha_1,\omega,\omega_1,\mu} f\right)(x) = \left(I_{0-}^{-\alpha,-\alpha_1,-\omega,-\omega_1,-\mu} f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{x,\infty}^{\alpha,-\alpha_1,-\omega+n,-\omega_1,-\mu+n} f\right)(x), \tag{16}$$

where $\Re(\mu) > 0$ and $n = [\Re(-\mu) + 1]$.

Lemma 1. *Let $\omega, \lambda, \beta, \rho \in c$, $x > 0$ be such that $\Re(\omega) > 0$, then*

$$\left(D_{+}^{\omega,\lambda,\beta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\beta+\omega+\lambda)}{\Gamma(\rho+\beta)\Gamma(\rho+\lambda)} x^{\rho+\lambda+1}, \tag{17}$$

where

$$(\Re(\rho) > -\min\{0, \Re(\omega+\lambda+\beta)\}). \tag{18}$$

Lemma 2. *Let $\omega, \lambda, \beta, \rho \in c$, $x > 0$ be such that $\Re(\omega) > 0$, then*

$$\left(D_{-}^{\omega,\lambda,\beta} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\lambda)\Gamma(1-\rho+\omega+\beta)}{\Gamma(1-\rho+\beta-\lambda)\Gamma(1-\rho)} x^{\rho+\lambda+1}, \tag{19}$$

where

$$(\Re(\rho-\sigma k) < 1 + \min\{\Re(-\lambda-n), \Re(\beta+\omega)\}, [\Re(\omega)] + 1). \tag{20}$$

Lemma 3. *Let $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ and $m = [\Re(\eta)] + 1$,*

$$\Re(\rho) - m > \max\left\{\begin{matrix} o, \Re(-\beta+\varepsilon) \\ \Re\left(-\beta-\beta'-\varepsilon'+\eta\right) \end{matrix}\right\}, \tag{21}$$

then the image will be

$$\left(D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho-\varepsilon+\beta)\Gamma\left(\beta+\beta'+\varepsilon'-\eta+\rho\right)}{\Gamma(-\varepsilon+\rho)\Gamma\left(\beta+\varepsilon'-\eta+\rho\right)\Gamma\left(\beta+\beta'-\eta+\rho\right)} x^{\beta+\beta'-\eta+\rho-1}. \tag{22}$$

Lemma 4. *Let $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ and $m = [\Re(\eta)] + 1$,*

$$\Re(\rho) + m > \max\left\{\begin{matrix} \Re\left(-\varepsilon'\right), \Re\left(\beta'+\varepsilon-\eta\right) \\ \Re\left(\beta+\beta'-\varepsilon'-\eta\right) + m \end{matrix}\right\}, \tag{23}$$

$$\left(D_{-}^{\beta,\beta,\varepsilon,\varepsilon,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma\left(1+\varepsilon'-\rho\right)\Gamma\left(1-\beta-\beta'+\eta-\rho\right)\Gamma\left(1-\beta'-\varepsilon+\eta-\rho\right)}{\Gamma(1-\rho)\Gamma\left(1-\beta'+\varepsilon'-\rho\right)\Gamma\left(1-\beta-\beta'-\varepsilon+\eta-\rho\right)} x^{\beta+\beta'-\eta+\rho-1}. \tag{24}$$

8. The Left-Sided MSM Fractional Differential Operator with Mittag-Leffler Function

Theorem 5. *Let $\omega, \lambda, \beta, \rho \in C$ $\Re > 0$ be such that $(\Re(\rho+\sigma k) > -\min\{0, \Re(\omega+\lambda+\beta)\})$, then the following result holds true:*

$$B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1} E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)\left(x(ts)^{\sigma}\right)\right)(z); p, q\right\} = z^{\rho+\nu-1}\Gamma(q) E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\left(xz^{\sigma}\right) *_4\Psi_3\left[\begin{matrix}\Delta \\ \Delta'\end{matrix}; \left(xz^{\sigma}\right)\right], \tag{25}$$

where

$$\begin{aligned} \Delta &= \{(p,\sigma), (\rho,\sigma), (\rho+\beta+\omega+\lambda,\sigma), (1,\sigma)\}, \\ \Delta' &= \{(p+q,\sigma), (\rho+\beta,\sigma), (\rho+\nu,\sigma)\}. \end{aligned} \tag{26}$$

Proof.

$$\begin{aligned} &B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1} E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)\left(x(ts)^{\sigma}\right)\right)(z); p, q\right\} \\ &= \int_0^1 s^{p+\sigma k-1}(1-s)^{q-1}\left(D_{+}^{\omega,\lambda,\beta}\left[t^{\rho-1} E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right]\left(xt^{\sigma}\right)\right)(z) \\ &= \int_0^1 s^{p+\sigma k-1}(1-s)^{q-1} \sum_{k=o}^{\infty} \frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k+\mu)} \\ &\quad \times \left(D_{+}^{\omega,\lambda,\beta} t^{\rho+\sigma k-1}\right)(x). \end{aligned} \tag{27}$$

By using the definition of beta function (Equation (13)), by using Lemma (Equation (17)), and changing ρ by $\rho+\sigma k$, we get the following:

$$\begin{aligned} &B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1} E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)\left(x(ts)^{\sigma}\right)\right)(z); p, q\right\} \\ &= \sum_{k=o}^{\infty} \frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k+\mu)} \\ &\quad \times \frac{\Gamma(p+\sigma k)\Gamma(q)\Gamma(\rho+\sigma k)\Gamma(\rho+\sigma k+\beta+\omega+\lambda)}{\Gamma(p+q+\sigma k)\Gamma(\rho+\sigma k+\beta)\Gamma(\rho+\sigma k+\lambda)} z^{\rho+\sigma k+\lambda-1}. \end{aligned} \tag{28}$$

By using Hadamard product (Equation (11)), we get the following:

$$B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1} E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)\left(x(ts)^{\sigma}\right)\right)(z); p, q\right\} = Z^{\rho+\lambda-1}\Gamma(q) E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\left(xz^{\sigma}\right) *_4\Psi_3\left[\begin{matrix}\Delta \\ \Delta'\end{matrix}; xt^{\sigma}\right]. \tag{29}$$

Corollary 6. *Let $\omega, \lambda, \beta, \rho \in C$ $\Re > 0$ be such that $(\Re(\rho+\sigma k) > -\min\{0, \Re(\omega+\lambda+\beta)\})$. Under the stated conditions, the* *right-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\gamma}\right)(x(ts)^{\sigma})\right)(z)\,;p,q\right\} = z^{\rho+\lambda-1}\Gamma(q)E_{\varepsilon,\mu}^{\gamma}(xz^{\sigma}) *{}_{4}\Psi_{3}\left[\begin{matrix}\Delta\\ \\ \Delta'\end{matrix}\,;(xz^{\sigma})\right], \tag{30}$$

where

$$\Delta = \{(p,\sigma),(\rho,\sigma),(\rho+\beta+\omega+\lambda,\sigma),(1,\sigma)\},\quad \Delta' = \{(p+q,\sigma),(\rho+\beta,\sigma),(\rho+\lambda,\sigma)\}. \tag{31}$$

Select a bounded sequence $k_n = 1$ and then proceed (Equation (30)).

Corollary 7. *Let $\omega,\lambda,\beta,\rho \in C\,\Re > 0$, such that $(\Re(\rho+\sigma k) > -\min\{0,\Re(\omega+\lambda+\beta)\})$. Under the stated conditions, the right-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{\left(D_{+}^{\omega,\lambda,\beta}\left(t^{\rho-1}\Phi_{p}^{(\{k_n\}_{n\in N_0})}\right)(\gamma\,;1\,;x(ts)^{\sigma})\right)(z)\,;p,q\right\} = z^{\rho+\lambda-1}\Gamma(q)\Phi_{p}^{(\{k_n\}_{n\in N_0})}(\gamma\,;1\,;xz^{\sigma}) *{}_{4}\Psi_{3}\left[\begin{matrix}\Delta\\ \\ \Delta'\end{matrix}\,;(xz^{\sigma})\right], \tag{32}$$

where

$$\Delta = \{(p,\sigma),(\rho,\sigma),(\rho+\beta+\omega+\lambda,\sigma),(1,\sigma)\},\quad \Delta' = \{(p+q,\sigma),(\rho+\beta,\sigma),(\rho+\lambda,\sigma)\}. \tag{33}$$

If we select $\xi=\mu=1$, then an extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

Theorem 8. *Let $\beta,\beta',\varepsilon,\varepsilon',\eta,\rho \in C$ and $m = [\Re(\eta)]+1$,*

$$\Re(\rho) - m > \max\left\{\begin{matrix}0,\Re(-\beta+\varepsilon)\\ \Re\left(-\beta-\beta'-\varepsilon'+\eta\right)\end{matrix}\right\}, \tag{34}$$

then

$$B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](x(ts)^{\sigma})(z)\,;p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(q)E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}(xz^{\sigma}) *{}_{5}\Psi_{4}\left[\begin{matrix}\Delta\\ \\ \Delta'\end{matrix}\,;x(z/s)^{\sigma}\right], \tag{35}$$

where

$$\Delta = \left\{(\rho,\sigma),(\beta-\varepsilon+\rho,\sigma),\left(\beta+\beta'+\varepsilon'-\eta+\rho,\sigma\right),(1,\sigma),(p,\sigma)\right\},$$
$$\Delta' = \left\{(-\varepsilon+\rho,\sigma),\left(\beta+\beta'-\eta+\rho,\sigma\right),\left(\beta+\varepsilon'-\eta+\rho,\sigma\right),(p+q,\sigma)\right\}. \tag{36}$$

Proof.

$$\begin{aligned}
&B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](x(ts)^{\sigma})(z)\,;p,q\right\}\\
&= \int_0^1 s^{p+\sigma k-1}(1-s)^{q-1}\left(D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](xt^{\sigma})\right)(z)\\
&= \int_0^1 s^{p+\sigma k-1}(1-s)^{q-1}ds\sum_{k=o}^{\infty}\frac{B_p^{(\{k_n\}_{n\in N_0})}(r+k,1-\gamma\,;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\\
&\quad\times\left(D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}t^{\rho-1}\right)(x).
\end{aligned} \tag{37}$$

By using the definition of beta function (Equation (13)) and by using lemma (Equation (22)), we get the following:

$$\begin{aligned}
&B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](x(ts)^{\sigma})(z)\,;p,q\right\}\\
&= \sum_{k=o}^{\infty}\frac{B_p^{(\{k_n\}_{n\in N_0})}(r+k,1-\gamma\,;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\times\frac{\Gamma(p+\sigma k)\Gamma(q)}{\Gamma(p+q+\sigma k)}\\
&\quad\cdot\frac{\Gamma(\rho)\Gamma(\rho-\varepsilon+\beta)\Gamma\left(\beta+\beta'+\varepsilon'-\eta+\rho\right)}{\Gamma(-\varepsilon+\rho)\Gamma\left(\beta+\varepsilon'-\eta+\rho\right)\Gamma\left(\beta+\beta'-\eta+\rho\right)}\times x^{\beta+\beta'-\eta+\rho-1}.
\end{aligned} \tag{38}$$

By changing ρ by $\rho+\sigma k$,

$$\begin{aligned}
&B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](x(ts)^{\sigma})(z)\,;p,q\right\}\\
&= \sum_{k=o}^{\infty}\frac{B_p^{(\{k_n\}_{n\in N_0})}(r+k,1-\gamma\,;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\times\frac{\Gamma(p+\sigma k)\Gamma(q)}{\Gamma(p+q+\sigma k)}\\
&\quad\times\frac{\Gamma(\rho+\sigma k)(\Gamma\rho+\sigma k-\varepsilon+\beta)\Gamma\left(\beta+\beta'+\varepsilon'-\eta+\rho+\sigma k\right)}{\Gamma(-\varepsilon+\rho+\sigma k)\Gamma\left(\beta+\varepsilon'-\eta+\rho+\sigma k\right)\Gamma\left(\beta+\beta'-\eta+\rho+\sigma k\right)}\\
&\quad\times Z^{\beta+\beta'-\eta+\rho+\sigma k-1}.
\end{aligned} \tag{39}$$

By using the Hadamard product which is given in (Equation (11)), we get the following:

$$B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}\right](x(ts)^{\sigma})(z)\,;p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(q)E_{\varepsilon,\mu}^{(\{k_n\}_{n\in N_0};\gamma)}(xt^{\sigma}) *{}_{5}\Psi_{4}\left[\begin{matrix}\Delta\\ \\ \Delta'\end{matrix}\,;xz^{\sigma}\right]. \tag{40}$$

Corollary 9. *Let the parameters* $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ *and* $m = [\Re(\eta)] + 1$, *and under the stated conditions, the left-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\gamma}\right](x(ts)^{\sigma})(z);p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(q)E_{\varepsilon,\mu}^{\gamma}(xz^{\sigma})*_5\Psi_4\left[\begin{matrix}\Delta\\ \Delta'\end{matrix};xz^{\sigma}\right], \quad (41)$$

where

$$\Delta = \left\{(\rho,\sigma),(-\varepsilon+\rho+\beta,\sigma),\left(\beta+\beta'+\varepsilon'-\eta+\rho,\sigma\right),(1,\sigma),(p,\sigma)\right\},$$
$$\Delta' = \left\{(-\varepsilon+\rho,\sigma),\left(\beta+\beta'-\eta+\rho,\sigma\right),\left(\beta+\varepsilon'-\eta+\rho,\sigma\right),(p+q,\sigma)\right\}. \quad (42)$$

Select a bounded sequence $k_n = 1$*and then proceed (Equation (41)).*

Corollary 10. *Let the parameters*$\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ *and* $m = [\Re(\eta)] + 1$, *and under the stated conditions, the left-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{\left(D_{+}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left(t^{\rho-1}\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}(\gamma;1;x(ts)^{\sigma})(z);p,q\right\} = z^{\beta+\beta'-\eta+\rho-1}\Gamma(q)E_{\varepsilon,\mu}^{\gamma}\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}(\gamma;1;xz^{\sigma})*_5\Psi_4\left[\begin{matrix}\Delta\\ \Delta'\end{matrix};x(z/s)^{\sigma}\right], \quad (43)$$

where

$$\Delta = \left\{(\rho,\sigma),(\beta-\varepsilon+\rho,\sigma),\left(\beta+\beta'+\varepsilon'-\eta+\rho,\sigma\right),(1,\sigma),(p,\sigma)\right\},$$
$$\Delta' = \left\{(-\varepsilon+\rho,\sigma),\left(\beta+\beta'-\eta+\rho,\sigma\right),\left(\beta+\varepsilon'-\eta+\rho,\sigma\right),(p+q,\sigma)\right\}. \quad (44)$$

If we select $\xi = \mu = 1$, *then an extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.*

9. The Right-Sided MSM Fractional Differential Operator with Mittag-Leffler Function

Theorem 11. *Let* $\omega, \lambda, \beta, \rho \in C\,\Re > 0$ *where* $(\Re(\rho-\sigma k) < 1 + \min\{\Re(-\lambda-n), \Re(\beta+\omega)\}$ *and* $n = [\Re(\omega)] + 1$, *then the following result holds true:*

$$B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)(x(t/s)^{-\sigma})\right)(z);p,q\right\} = z^{\rho+\lambda+1}\Gamma(-q+2)E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(xz^{-\sigma})*_4\Psi_3\left[\begin{matrix}\Delta\\ \Delta'\end{matrix};xz^{-\sigma}\right], \quad (45)$$

where

$$\Delta = \{(-p+2,\sigma),(1-\rho-\lambda,\sigma),(1-\rho+\omega+\beta,\sigma),(1,\sigma)\},$$
$$\Delta' = \{(-p-q+4,\sigma),(1-\rho+\beta-\lambda,\sigma),(1-\rho)\}. \quad (46)$$

Proof.

$$\begin{aligned}
&B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)(x(t/s)^{-\sigma})\right)(z);p,q\right\}\\
&= \int_0^1 s^{-(p-\sigma k-1)}(1-s)^{-(q-1)}\left(D_{-}^{\omega,\lambda,\beta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t)^{-\sigma})\right)(z),\\
&= \int_0^1 s^{-p+\sigma k+1}(1-s)^{-q+1}\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\\
&\times\left(D_{-}^{\omega,\lambda,\beta}t^{\rho-\sigma k-1}\right)(x), = \int_0^1 s^{-p+\sigma k+2-1}(1-s)^{-q+2-1}\\
&\cdot\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\times\left(D_{-}^{\omega,\lambda,\beta}t^{\rho-\sigma k-1}\right)(x).
\end{aligned} \quad (47)$$

By using the definition of beta function (Equation (13)), by using lemma (Equation (19)), and changing ρ by $\rho-\sigma k$, we get the following:

$$\begin{aligned}
&B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)(x(t/s)^{-\sigma})\right)(z);p,q\right\}\\
&= \sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\\
&\times\frac{\Gamma(-p+2+\sigma k)\Gamma(-q+2)\Gamma(1-\rho+\sigma k-\lambda)\Gamma(1-\rho+\sigma k+\omega+\lambda)}{\Gamma(-p-q+4+\sigma k)\Gamma(1-\rho+\sigma k+\beta-\lambda)\Gamma(1-\rho+\sigma k)}z^{\rho+\lambda+1}.
\end{aligned} \quad (48)$$

By using Hadamard product (Equation (11)), we get the following:

$$B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right)(x(t/s)^{-\sigma})\right)(z);p,q\right\} = Z^{\rho+\lambda+1}\Gamma(-q+2)E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(xt^{-\sigma})*_4\Psi_3\left[\begin{matrix}\Delta\\ \Delta'\end{matrix};xt^{-\sigma}\right]. \quad (49)$$

Corollary 12. *Let* $\omega, \lambda, \beta, \rho \in C\,\Re > 0$*where* $(\Re(\rho-\sigma k) < 1 + \min\{\Re(-\lambda-n), \Re(\beta+\omega)\}, [\Re(\omega)] + 1)$, *under the stated conditions, the left-sided Caputo fractional differential* *operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}E_{\varepsilon,\mu}^{\gamma}\right)\left(x(t/s)^{-\sigma}\right)\right)(z);p,q\right\} = z_{-}^{\omega,\lambda,\beta}\Gamma(-q+2)E_{\varepsilon,\mu}^{\gamma}(xz^{-\sigma})*{}_4\Psi_3\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{-\sigma}\right], \quad (50)$$

where

$$\Delta=\{(-p+2,\sigma),(1-\rho-\lambda,\sigma),(1-\rho+\beta+\omega+\lambda,\sigma),(1,\sigma)\},$$
$$\Delta'=\{(-p-q+4,\sigma),(1-\rho+\beta-\lambda,\sigma),(1-\rho,\sigma)\}. \quad (51)$$

Select a bounded sequence $k_n=1$ and then proceed (Equation (50)).

Corollary 13. *Let $\omega,\lambda,\beta,\rho\in C\,\Re>0$ where $(\Re(\rho-\sigma k)<1+\min\{\Re(-\lambda-n),\Re(\beta+\omega)\},[\Re(\omega)]+1)$; under the stated conditions, the left-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{\left(D_{-}^{\omega,\lambda,\beta}\left(t^{\rho-1}\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}\right)(\gamma;1;x(t/s)^{-\sigma})\right)(z);p,q\right\} = z^{\omega,\lambda,\beta}\Gamma(-q+2)\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}(\gamma;1;xz^{-\sigma})*{}_4\Psi_3\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{-\sigma}\right], \quad (52)$$

where

$$\Delta=\{(-p+2,\sigma),(1-\rho-\lambda,\sigma),(1-\rho+\beta+\omega+\lambda,\sigma),(1,\sigma)\},$$
$$\Delta'=\{(-p-q+4,\sigma),(1-\rho+\beta-\lambda,\sigma),(1-\rho,\sigma)\}. \quad (53)$$

If we select $\xi=\mu=1$, then an extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

Theorem 14. *${}_{Let}\beta,\beta',\varepsilon,\varepsilon',\eta,\rho\in C$ and $m=[\Re(\eta)]+1$ be such that*

$$\Re(\rho)+m>\max\left\{\begin{matrix}\Re\left(-\varepsilon'\right),\Re\left(\beta'+\varepsilon-\eta\right) \\ \Re\left(\beta+\beta'-\eta\right)+m)\end{matrix}\right\}, \quad (54)$$

then

$$B\left\{D_{-}^{\beta,\beta,\varepsilon,\varepsilon,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t/s)^{-\sigma})(z);p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(-q+2)E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(xz^{-\sigma})*{}_5\Psi_4\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{-\sigma}\right], \quad (55)$$

where

$$\Delta=\left\{\left(1+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'+\eta-\rho,\sigma\right),\left(1-\beta'-\varepsilon+\eta-\rho,\sigma\right),(1,\sigma),(-p+2,\sigma)\right\},$$
$$\Delta'=\left\{(1-\rho,\sigma),\left(1-\beta'+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'-\varepsilon+\eta-\rho,\sigma\right),(-p-q+4,\sigma)\right\}. \quad (56)$$

Proof.

$$\begin{aligned}
&B\left\{D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t/s)^{-\sigma})(z);p,q\right\} \\
&=\int_0^1 s^{-(p-\sigma k-1)}(1-s)^{-(q-1)}\left(D_{-}^{\beta,\beta',\varepsilon,\varepsilon'+\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right]\left(x(t)^{-\sigma}\right)\right)(z), \\
&=\int_0^1 s^{-p+\sigma k+1}(1-s)^{-q+1}\left(D_{-}^{\beta,\beta',\varepsilon,\varepsilon'+\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right]\left(x(t)^{-\sigma}\right)\right)(z), \\
&=\int_0^1 s^{-p+\sigma k+1}(1-s)^{-q+1}\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)} \\
&\times\left(D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}t^{\rho-\sigma k-1}\right)(x),=\int_0^1 s^{-p+\sigma k+2-1}(1-s)^{-q+2-1} \\
&\cdot\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\times\left(D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}t^{\rho-\sigma k-1}\right)(x).
\end{aligned} \quad (57)$$

By using a definition of beta function (Equation (13)) and by using lemma (Equation (24)), we get the following:

$$\begin{aligned}
&B\left\{D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t/s)^{-\sigma})(z);p,q\right\} \\
&=\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\varepsilon k+\mu)}\times\frac{\Gamma(-p+2+\sigma k)\Gamma(-q+2)}{\Gamma(-p-q+4+\sigma k)} \\
&\times\frac{\Gamma\left(1+\varepsilon'-\rho\right)\left(\Gamma1-\beta-\beta'+\eta-\rho\right)\Gamma\left(1-\beta'-\varepsilon+\eta-\rho\right)}{\Gamma(1-\rho)\Gamma\left(1-\beta'+\varepsilon'-\rho\right)\Gamma\left(1-\beta-\beta'-\varepsilon+\eta-\rho\right)}x^{\beta+\beta'-\eta+\rho-1}.
\end{aligned} \quad (58)$$

By putting $\rho=\rho-\sigma k$,

$$\begin{aligned}
&B\left\{D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t/s)^{-\sigma})(z);p,q\right\} \\
&=\sum_{k=o}^{\infty}\frac{B_p^{\left(\{k_n\}_{n\in N_0}\right)}(r+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\sqrt{(\varepsilon k+\mu)}}\times\frac{\Gamma(-p+2+\sigma k)\Gamma(-q+2)}{\Gamma(-p-q+4+\sigma k)} \\
&\times\frac{\Gamma\left(1+\varepsilon'-\rho+\sigma k\right)\left(\Gamma1-\beta-\beta'+\eta-\rho+\sigma k\right)\Gamma\left(1-\beta'-\varepsilon+\eta-\rho+\sigma k\right)}{\Gamma(1-\rho+\sigma k)\Gamma\left(1-\beta'+\varepsilon'-\rho+\sigma k\right)\Gamma\left(1-\beta-\beta'-\varepsilon+\eta-\rho+\sigma k\right)} \\
&\times x^{\beta+\beta'-\eta+\rho-\sigma k-1}.
\end{aligned} \quad (59)$$

By Hadamard product (Equation (11)), we get the following:

$$B\left\{D_{-}^{\beta,\beta',\varepsilon,\varepsilon',\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}\right](x(t/s)^{-\sigma})(z);p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(-q+2)E_{\varepsilon,\mu}^{\left(\{k_n\}_{n\in N_0};\gamma\right)}(xt^{\sigma})*{}_5\Psi_4\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{\sigma}\right]. \quad (60)$$

Corollary 15. *Let the parameters* $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ *and* $m = [\Re(\eta)] + 1$*, and under the stated conditions, the right-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{D_{-}^{\beta,\beta\prime,\varepsilon,\varepsilon\prime,\eta}\left[t^{\rho-1}E_{\varepsilon,\mu}^{\gamma}\right](x(t/s)^{-\sigma})(z)\,;p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(-q+2)E_{\varepsilon,\mu}^{\gamma}(xz^{-\sigma}) * {}_5\Psi_4\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{-\sigma}\right], \tag{61}$$

where

$$\Delta = \left\{\left(1+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'+\eta-\rho,\sigma\right),\left(1-\beta'-\varepsilon+\eta-\rho,\sigma\right),(1,\sigma),(p,\sigma)\right\},$$
$$\Delta' = \left\{(1-\rho,\sigma),\left(1-\beta'+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'-\varepsilon+\eta-\rho,\sigma\right),(p+q,\sigma)\right\}. \tag{62}$$

Select a bounded sequence $k_n = 1$ *and then proceed (Equation (61)).*

Corollary 16. *Let the parameters* $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ *and* $m = [\Re(\eta)] + 1$*, and under the stated conditions, the right-sided Caputo fractional differential operator of extended Mittag-Leffler function is defined as follows:*

$$B\left\{D_{-}^{\beta,\beta\prime,\varepsilon,\varepsilon\prime,\eta}\left[t^{\rho-1}\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}\right](\gamma\,;1\,;x(t/s)^{-\sigma})(z)\,;p,q\right\} = Z^{\beta+\beta'-\eta+\rho-1}\Gamma(-q+2)\Phi_p^{\left(\{k_n\}_{n\in N_0}\right)}(\gamma\,;1\,;xz^{-\sigma}) * {}_5\Psi_4\left[\begin{matrix}\Delta \\ \Delta'\end{matrix};xz^{-\sigma}\right], \tag{63}$$

where

$$\Delta = \left\{\left(1+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'+\eta-\rho,\sigma\right),\left(1-\beta'-\varepsilon+\eta-\rho,\sigma\right),(1,\sigma),(p,\sigma)\right\},$$
$$\Delta' = \left\{(1-\rho,\sigma),\left(1-\beta'+\varepsilon'-\rho,\sigma\right),\left(1-\beta-\beta'-\varepsilon+\eta-\rho,\sigma\right),(p+q,\sigma)\right\}. \tag{64}$$

If we select $\xi = \mu = 1$*, then an extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.*

Remark 17. In this paper, beta operator was applied on Caputo MSM fractional differentiation of extended Mittag-Leffler function. New results and some corollaries had been demonstrated. Above corollaries can easily be derived if we select $\xi = \mu = 1$, and then, the above results reduce for classical confluent hypergeometric functions.

10. Conclusion

We accomplish our current consideration by commenting further the outcomes acquired here by the beta operator with Caputo MSM fractional differentiation of extended Mittag-Leffler function. It is to be noted that the results in our study will sufficiently be significant, most generally in nature and capable for the differential transform techniques with the numerous special functions through appropriate selections by means of arbitrary parameters that will be elaborated in these consequences. So, the outcomes existing in our exploration will be relied upon leading some potential application in various fields like numerical, physical, measurable, and design sciences. Differential operators are very useful for solving problems. In many fields of applied sciences, especially in extended form of functions like beta function, gamma function, Gauss hypergeometric function, confluent hypergeometric function, and Mittag-Leffler function, the Mittag-Leffler function rises certainly in the solution of fractional-order integral equations and in the examinations of the fractional generalization of the kinetic equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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