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Research Article

The Soliton Solutions and Long-Time Asymptotic Analysis for a General Coupled KdV Equation

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A general coupled KdV equation, which describes the interactions of two long waves with different dispersion relation, is considered. By employing the Hirota's bilinear method, the bilinear form is obtained, and the one-soliton solution and two-soliton solution are constructed. Moreover, the elasticity of the collision between two solitons is proved by analyzing the asymptotic behavior of the two-soliton solution. Some figures are displayed to illustrate the process of elastic collision.

1. Introduction

In the soliton theory, finding the soliton solutions for the nonlinear partial equations is becoming more and more important since the soliton solutions can describe many complex physical phenomena [1]. Many effective approaches have been proposed, such as the inverse scattering transformation method [2], the Bäcklund transformation method [3], the Darboux transformation method [4–6], the Hirota bilinear method [7–11], and the Riemann-Hilbert method [12]. Among them, the Hirota bilinear method is not only direct but also effective for investigating the soliton solutions.

In the past decades, the coupled Korteweg-de Vries (KdV) equations have been investigated widely and many integrable coupled KdV equations are found. For example, Gurses and Karasu [13] showed that the following coupled KdV equation was integrable and admitted recursion operator and a bi-Hamiltonian structure:

$$\begin{cases} u_t + u_{xxx} - 6uu_x - 6v_x = 0, \\ v_t - 2v_{xxx} + 6\,uv_x = 0. \end{cases} \tag{1}$$

In fact, this equation is Lax integrable, and the Lax pair was firstly given by Drinfeld and Sokolov [14] and then by Bogoyavlenskii [15] and Karasu and Yurduşen [16] independently. Subsequently, this equation was also derived by Satsuma and Hirota [17] as one case of the four-reduction of the KP Hierarchy. Moreover, Karasu and Yurduşen [16] proposed a Bäcklund transformation and some explicit solutions of Equation (1). As far as we know, the soliton solutions and the collision between two solitons have not been investigated. So in this paper, we investigate the following general coupled KdV equation:

$$\begin{cases} u_t + u_{xxx} - 6uu_x + \alpha v_x = 0, \\ v_t - 2v_{xxx} + 6\,uv_x = 0, \end{cases} \tag{2}$$

which is just the coupled KdV Equation (1) for $\alpha = -6$.

The rest of this paper is organized as follows. In "The Bilinear Form and Soliton Solutions,", the bilinear form, the one-soliton, and two-soliton of Equation (2) are obtained based on the Hirota's direct method. In "Asymptotic Analysis on Two-Soliton Solution," the asymptotic behaviors are studied to prove that the two-soliton collision is elastic. Finally, conclusions are given in "Conclusion."

2. The Bilinear Form and Soliton Solutions

We implement the following dependent variable transformation to Equation (2):

$$\begin{cases} u = -2[\log(\phi)]_{xx}, \\ v = \dfrac{g}{\phi}, \end{cases} \tag{3}$$

where g and ϕ are functions of x and t. Then, the following bilinear equations of Equation (2) are obtained as follows:

$$\begin{cases} \left(D_x D_t + D_x^4\right)\phi \cdot \phi = \alpha \phi g, \\ \left(D_t - 2D_x^3\right) g \cdot \phi = 0, \end{cases} \tag{4}$$

where the D-operators [18] are defined by

$$D_x^m D_t^n a(x,t) \cdot b(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t) b\left(x', t'\right)\Big|_{x=x',t=t'}, \tag{5}$$

where both m and n are integers.

In order to apply the perturbation method to Equation (4) to find the soliton solutions of Equation (2), we expand functions g and ϕ in power series of a small parameter ε as

$$g = \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots, \tag{6}$$

$$\phi = 1 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \cdots, \tag{7}$$

where g_i and $\phi_i (i = 1, 2, 3, \cdots)$ are functions of x and t. Substituting Equations (6) and (7) into the bilinear Equation (4) and collecting the coefficients of parameter ε, we have

$$\begin{aligned} \varepsilon^1 &: 2(\phi_{1xt} + \phi_{1xxxx}) = \alpha g_1, \\ & g_{1t} - 2g_{1xxx} = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} \varepsilon^2 &: 2(\phi_{2xt} + \phi_{2xxxx}) = -\left(D_x D_t + D_x^4\right)\phi_1 \cdot \phi_1 + \alpha(g_2 + \phi_1 g_1), \\ & g_{2t} - 2g_{2xxx} = -\left(D_t - 2D_x^3\right) g_1 \cdot \phi_1, \end{aligned} \tag{9}$$

$$\begin{aligned} \varepsilon^3 &: 2(\phi_{3xt} + \phi_{3xxxx}) = -\left(D_x D_t + D_x^4\right)(\phi_1 \cdot \phi_2 + \phi_2 \cdot \phi_1) \\ & \quad + \alpha(g_3 + \phi_1 g_2 + \phi_2 g_1), \\ & g_{3t} - 2g_{3xxx} = -\left(D_t - 2D_x^3\right)(g_1 \cdot \phi_2 + g_2 \cdot \phi_1), \end{aligned} \tag{10}$$

$$\begin{aligned} \varepsilon^4 &: 2(\phi_{4xt} + \phi_{4xxxx}) = -\left(D_x D_t + D_x^4\right)(\phi_1 \cdot \phi_3 + \phi_2 \cdot \phi_2 + \phi_3 \cdot \phi_1) \\ & \quad + \alpha(g_4 + \phi_1 g_3 + \phi_2 g_2 + \phi_3 g_1), \\ & g_{4t} - 2g_{4xxx} = -\left(D_t - 2D_x^3\right)(g_1 \cdot \phi_3 + g_2 \cdot \phi_2 + g_3 \cdot \phi_1), \end{aligned} \tag{11}$$

$$\begin{aligned} \varepsilon^5 &: 2(\phi_{5xt} + \phi_{5xxxx}) = -\left(D_x D_t + D_x^4\right)(\phi_1 \cdot \phi_4 + \phi_2 \cdot \phi_3 \\ & \quad + \phi_3 \cdot \phi_2 + \phi_4 \cdot \phi_1) + \alpha(g_5 + \phi_1 g_4 \\ & \quad + \phi_2 g_3 + \phi_3 g_2 + \phi_4 g_1), \\ & g_{5t} - 2g_{5xxx} = -\left(D_t - 2D_x^3\right)(g_1 \cdot \phi_4 + g_2 \cdot \phi_3 + g_3 \\ & \quad \cdot \phi_2 + g_4 \cdot \phi_1). \end{aligned} \tag{12}$$

2.1. One-Soliton Solution. To obtain the one-soliton solution for the general coupled KdV Equation (2), set

$$g_1 = e^{\eta_1}, \tag{13}$$

where $\eta_1 = k_1 x + \omega_1 t + \delta_1$. Substituting it into Equation (8), we have $\omega_1 = 2{k_1}^3$ and

$$\phi_1 = b_1 e^{\eta_1}, b_1 = \frac{1}{6}\frac{\alpha}{{k_1}^4}. \tag{14}$$

Furthermore, from Equation (9), we have

$$\begin{aligned} & g_2 = 0, \\ & \phi_2 = b_{11} e^{2\eta_1}, b_{11} = \frac{1}{288}\frac{\alpha^2}{{k_1}^8}. \end{aligned} \tag{15}$$

Assuming $g_j = \phi_j = 0, (j = 3, 4, \cdots)$, it is easy to see that Equation (10) and other equations from the coefficients of parameter ε are satisfied automatically. So we obtain the following one-soliton solution for the general coupled KdV Equation (2) by setting $\varepsilon = 1$:

$$\begin{cases} u = -2[\log(1 + \phi_1 + \phi_2)]_{xx}, \\ v = \dfrac{g_1}{1 + \phi_1 + \phi_2}. \end{cases} \tag{16}$$

Figures 1(a) and 1(b) demonstrate the soliton structures of one-solutions $u(x,t)$ and $v(x,t)$, respectively, for parameters $k_1 = 1/2, \alpha = 1$.

2.2. Two-Soliton Solution. Likewise, to arrive the two-soliton solution, we set

$$g_1 = e^{\eta_1} + e^{\eta_2}, \tag{17}$$

where $\eta_i = k_i x + \omega_i t + \delta_i$, $i = 1, 2$. Plugging Equation (17) into Equation (8) yields $\omega_i = 2{k_i}^3$, $i = 1, 2$, and

$$\phi_1 = b_1 e^{\eta_1} + b_2 e^{\eta_2}, \tag{18}$$

with $b_i = 1/6\,(\alpha/{k_i}^4)$, $i = 1, 2$.

From Equations (9) and (18), we have

$$\begin{aligned} & g_2 = a_{12} e^{\eta_1 + \eta_2}, \\ & \phi_2 = b_{11} e^{2\eta_1} + b_{22} e^{2\eta_2} + b_{12} e^{\eta_1 + \eta_2}, \end{aligned} \tag{19}$$

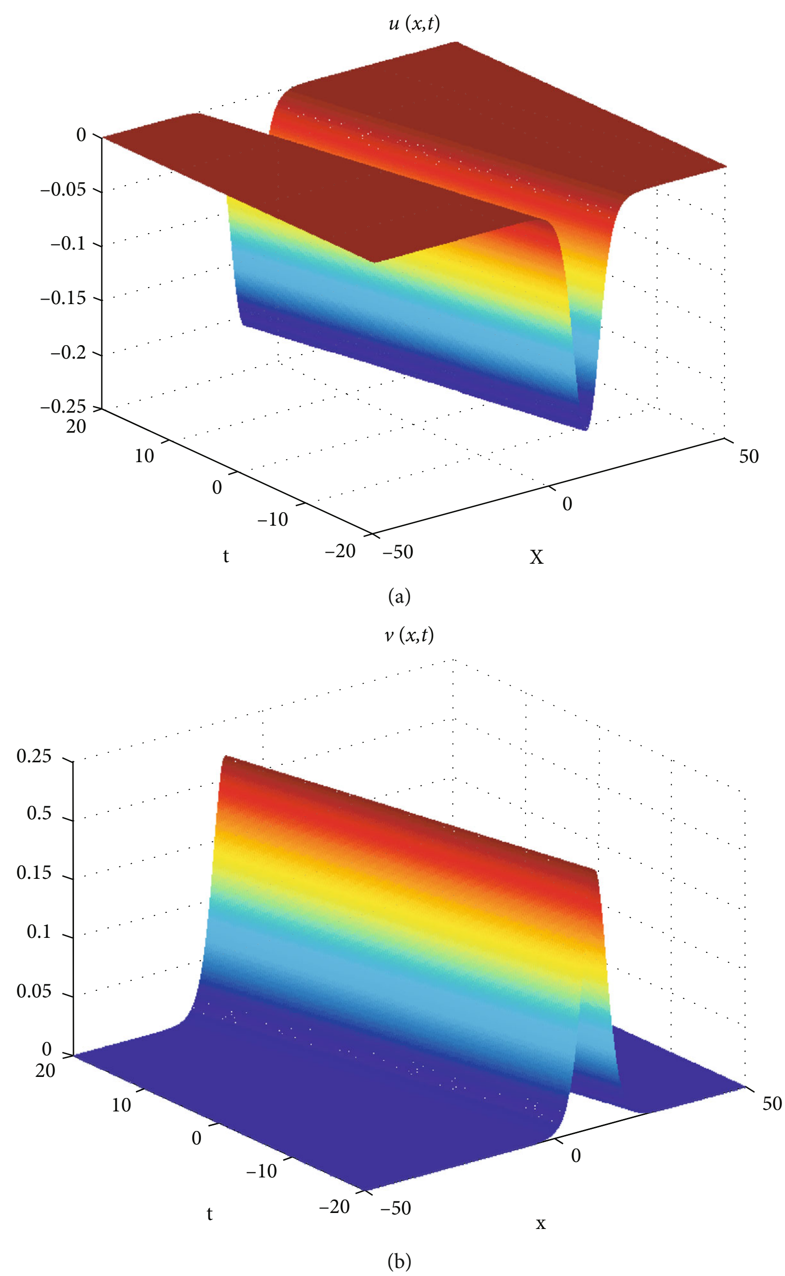

FIGURE 1: Time and space evolution of the antibell soliton solution $u(x,t)$ and bell soliton solution $v(x,t)$ in Equation (16) for parameters $k_1 = 1/2, \alpha = 1$. (a) $u(x,t)$. (b) $v(x,t)$.

with

$$a_{12} = \frac{1}{6}\,\frac{\alpha\left(k_1 - k_2\right)^2\left(k_1^2 - k_2^2\right)}{k_1^{\,4} k_2^{\,4}},$$

$$b_{11} = \frac{1}{288}\,\frac{\alpha^2}{k_1^{\,8}},\; b_{22} = \frac{1}{288}\,\frac{\alpha^2}{k_2^{\,8}},$$
$$b_{12} = \frac{1}{36}\,\frac{\alpha^2\left(k_1^4 + k_2^4\right)}{k_1^4 k_2^4\left(k_1^2 + k_2^2\right)\left(k_1 + k_2\right)^2}. \qquad (20)$$

Similarly, from Equations (10), (18), and (19), g_3 and ϕ_3 can be derived as

$$g_3 = a_{112} e^{2\eta_1 + \eta_2} + a_{122} e^{\eta_1 + 2\eta_2},$$
$$\phi_3 = b_{112} e^{2\eta_1 + \eta_2} + b_{122} e^{\eta_1 + 2\eta_2}, \qquad (21)$$

with

$$a_{112} = \frac{1}{288}\,\frac{\alpha^2\left(k_1 - k_2\right)^2}{k_1^8\left(k_1 + k_2\right)^2},$$

$$a_{122} = \frac{1}{288}\,\frac{\alpha^2\left(k_1 - k_2\right)^2}{k_2^8\left(k_1 + k_2\right)^2},$$

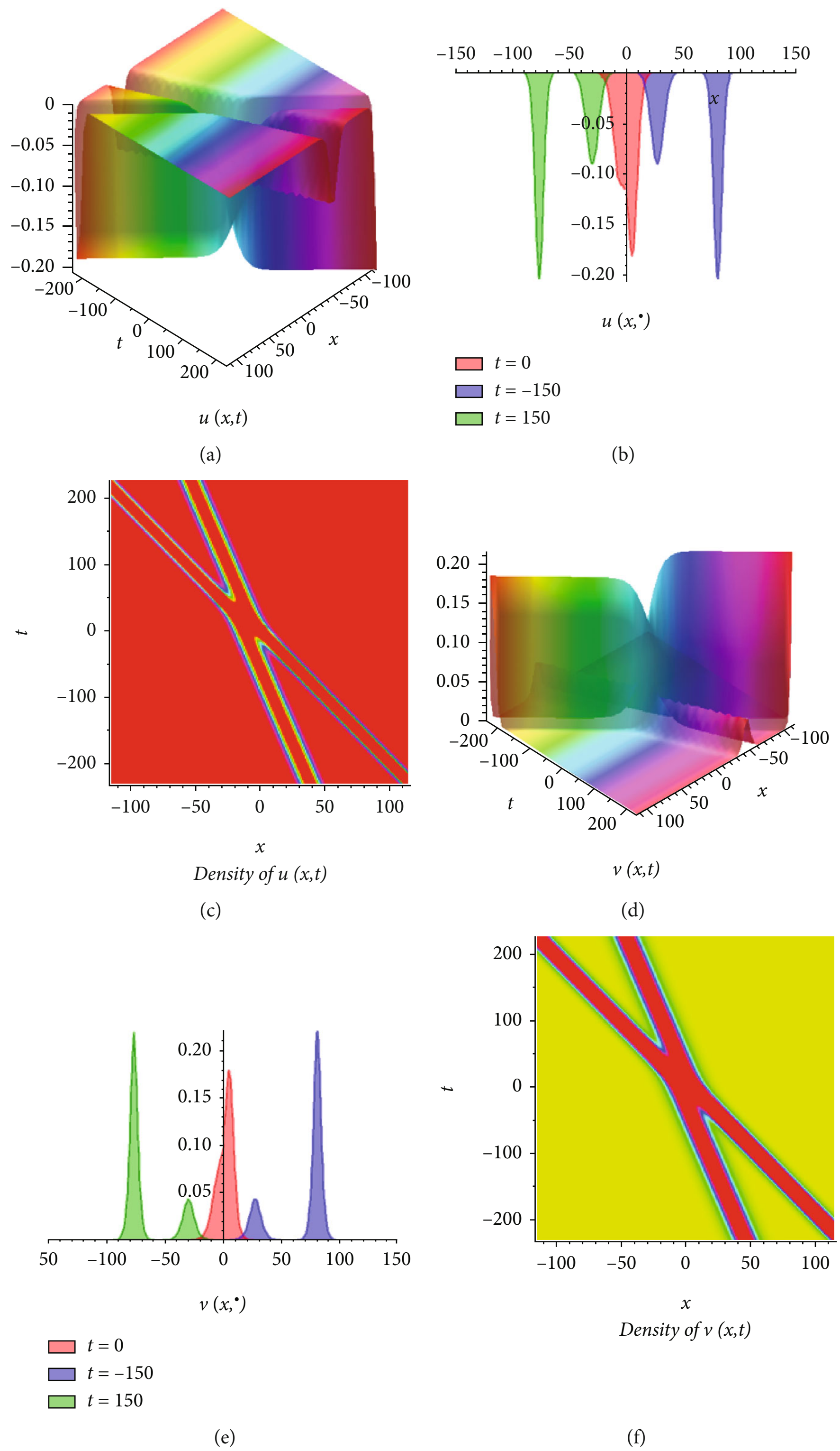

Figure 2: (a, d) Time evolution of the 2-soliton solutions u and v with parameters $k_1 = 1/2, k_2 = 1/3, \alpha = 1$, respectively. (b, e) Soliton interaction shots of u and v at $t = -150$, $t = 0$, and $t = 150$. Two solitons travel from right to left and pass through each other while their shapes well-maintained, implying perfect elasticity of the collision. (c, f) Density profile of the collision progress of u and v, showing the velocity keep invariable and phase shift after interaction. (a) $u(x, t)$. (b) $u(x, \cdot)$. (c) density of $u(x, t)$. (d) $v(x, t)$. (e) $v(x, \cdot)$. (f) density of $v(x, t)$.

$$b_{112} = \frac{1}{1728} \frac{\alpha^3 (k_1 - k_2)^2}{k_1^8 k_2^4 (k_1 + k_2)^2},$$
$$b_{122} = \frac{1}{1728} \frac{\alpha^3 (k_1 - k_2)^2}{k_1^4 k_2^8 (k_1 + k_2)^2}. \quad (22)$$

Moreover, if we plug the above obtained g_1, g_2, g_3 and ϕ_1, ϕ_2, ϕ_3 into Equation (11), we get

$$g_4 = 0,$$
$$\phi_4 = b_{1122} e^{2\eta_1 + 2\eta_2}, \quad (23)$$

with $b_{1122} = 1/82944\,(\alpha^4 (k_1 - k_2)^4 / k_1^8 k_2^8 (k_1 + k_2)^4)$.

Assuming $g_j = \phi_j = 0, (j = 5, 6, \cdots)$, it is observed that the left equations are satisfied automatically. By setting $\varepsilon = 1$, we obtain the two-soliton solutions for the coupled KdV Equation (2). We i4llustrate the structures of u and v in Figures 2(a) and 2(b).

3. Asymptotic Analysis on Two-Soliton Solution

Now we analyze the two-soliton solution of Equation (2) with long-time asymptotic method. Note that the two-soliton solution can be written as

$$u = -2[log(1 + \phi_1 + \phi_2 + \phi_3 + \phi_4)]_{xx},$$
$$v = \frac{g_1 + g_2 + g_3}{1 + \phi_1 + \phi_2 + \phi_3 + \phi_4}, \quad (24)$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ and g_1, g_2, g_3, g_4 are shown in Equations (17)–(23). Without loss of generality, suppose that $k_2 > k_1 > 0$.

For fixed η_1, note that $\eta_2 = (k_2/k1)\eta_1 + 2k_2(k_2^2 - k_1^2)t - (k_2/k_1)\delta_1 + \delta_2$, then arrive at the following:

(i) Solitons-1 before collision ($t \to -\infty, \eta_2 \sim -\infty$)

$$u_1 \sim -\frac{16 k_1^2 \beta_1 (\beta_1^2 + 4\beta_1 + 8)}{(\beta_1^2 + 8\beta_1 + 8)^2},$$
$$v_1 \sim \frac{8 e^{\eta_1}}{\beta_1^2 + 8\beta_1 + 8}, \quad (25)$$

where $\beta_1 = b_1 e^{\eta_1}$

(ii) Solitons-1 after collision ($t \to +\infty, \eta_2 \to +\infty$)

$$u_1 \sim -\frac{16 k_1^2 \beta_1' \left(\beta_1'^2 + 4\beta_1' + 8\right)}{\left(\beta_1'^2 + 8\beta_1' + 8\right)^2},$$
$$v_1 \sim \frac{8 e^{(\eta_1 + A_{12})}}{\beta_1'^2 + 8\beta_1' + 8}, \quad (26)$$

where $\beta_1' = b_1 e^{(\eta_1 + B_{12})}$, $A_{12} = a_{122}$, and $B_{12} = k_1 - k_2{}^2 / k_1 + k_2{}^2$.

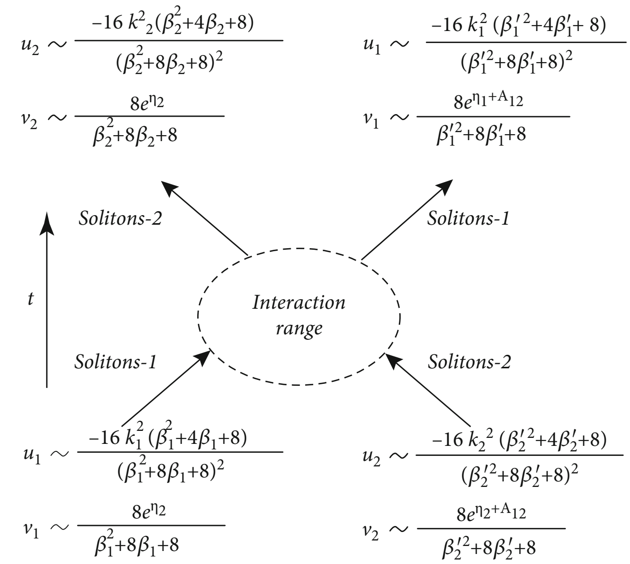

Figure 3: Two-soliton collisions in the coupled KdV system.

For fixed η_2, note that $\eta_1 = (k_1/k2)\eta_2 + 2k_1(k_1^2 - k_2^2)t - (k_1/k_2)\delta_2 + \delta_1$, and then we arrive at the following:

(i) ′ Solitons-2 before collision ($t \to -\infty, \eta_1 \to +\infty$)

$$u_2 \sim -\frac{16 k_2^2 \beta_2' \left(\beta_2'^2 + 4\beta_2' + 8\right)}{\left(\beta_2'^2 + 8\beta_2' + 8\right)^2},$$
$$v_2 \sim \frac{8 e^{(\eta_2 + A_{21})}}{\beta_2'^2 + 8\beta_2' + 8}, \quad (27)$$

where $\beta_2' = b_2 e^{(\eta_2 + B_{21})}$, $A_{21} = a_{112}$, $B_{21} = k_1 - k_2{}^2 / k_1 + k_2{}^2$

(ii) ′ Solitons-2 after collision ($t \to +\infty, \eta_1 \sim -\infty$)

$$u_2 \sim -\frac{16 k_2^2 \beta_2 (\beta_2^2 + 4\beta_2 + 8)}{(\beta_2^2 + 8\beta_2 + 8)^2},$$
$$v_2 \sim \frac{8 e^{\eta_2}}{\beta_2^2 + 8\beta_2 + 8}, \quad (28)$$

where $\beta_2 = b_2 e^{\eta_2}$.

The above asymptotic analysis can also be seen in Figure 3. Comparing the asymptotic expressions Equation (25) with Equation (26) and Equation (27) with Equation (28), we find that the amplitudes and velocities remain the same, but the phases are changed. To illustrate the collision process exactly, the graphs are presented in Figure 2, which shows that the collisions of the two-soliton waves are exactly elastic.

4. Conclusion

In conclusion, we studied a general coupled KdV Equation (2)via the Hirota's bilinear method. We first constructed the bilinear form and then the one-soliton solution and two-soliton solution. Furthermore, the asymptotic analysis is given to prove that the collision of the two-soliton solutions is elastic.

Data Availability

All data, models, and code generated or used during the study appear in the submitted article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Changhao Zhang is responsible for methodology, data curation, software, and validation. Guiying Chen is responsible for conceptualization, writing-original draft, visualization, writing (review and editing), supervision, and project administration. All authors have read and agreed to the published version of the manuscript.

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