

## Research Article

# Global Existence and Long-Time Behavior of Solutions to the Full Compressible Euler Equations with Damping and Heat Conduction in $\mathbb{R}^3$

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We study the Cauchy problem of the three-dimensional full compressible Euler equations with damping and heat conduction. We prove the existence and uniqueness of the global small  $H^N$  ( $N \geq 3$ ) solution; in particular, we only require that the  $H^4$  norms of the initial data be small when  $N \geq 5$ . Moreover, we use a pure energy method to show that the global solution converges to the constant equilibrium state with an optimal algebraic decay rate as time goes to infinity.

## 1. Introduction

We study the following Cauchy problem of the full compressible Euler equations with damping and heat conduction:

for  $(x, t) \in \mathbb{R}^3 \times [0, \infty)$ ,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = -\alpha \rho u, \\ (\rho \mathcal{E})_t + \operatorname{div}(\rho u \mathcal{E} + uP) = -\alpha \rho u^2 + \kappa \Delta \theta, \\ (\rho, u, \theta)(x, t)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \longrightarrow (1, 0, 1), |x| \longrightarrow \infty. \end{cases} \quad (1)$$

Here, the unknown variables  $\rho = \rho(x, t)$ ,  $u = u(x, t)$ ,  $\theta = \theta(x, t)$ ,  $P = P(x, t)$  denote the density, the velocity, the absolute temperature, and the pressure, respectively. The total energy per unit mass  $\mathcal{E} = (1/2)|u|^2 + e$ , and  $e$  is the internal energy per unit mass. The constants  $\alpha > 0$  and  $\kappa > 0$  are the friction damping coefficient and the thermal conductivity, respectively.

The system (1) can be used to model a compressible gas flow through a porous medium [1–3]. Assume that the gas is perfect and polytropic, then

$$P = R\rho\theta = Ae^{S/c_v}\rho^\gamma, \quad e = c_v\theta, \quad c_v = \frac{R}{\gamma - 1}, \quad (2)$$

where  $S$  is the entropy,  $R > 0$  and  $A > 0$  are the universal gas constants,  $\gamma > 1$  is the adiabatic exponent, and  $c_v > 0$  is the specific heat at constant volume.

We review the known results about the compressible Euler equations with damping. There are a lot of research works on the compressible isentropic Euler equations with damping in dimension one. For the Cauchy problems, readers can refer to [4, 5] for the existence of the global  $BV$  solutions, to [6–11] for the global  $L^\infty$  entropy-weak solutions with vacuum, and to [12, 13] for small smooth solutions. For the initial-boundary value problems, readers can refer to [14, 15] for the existence of the global  $L^\infty$  entropy-weak solutions and to [2, 16, 17] for small smooth solutions. For the asymptotic convergence of solutions, we refer to [8–11] for  $L^\infty$

entropy-weak solutions and to [13, 18–20] for small smooth solutions. In addition, there are some results on the 1D compressible nonisentropic Euler equations with damping (see [1–3, 21–23]). The global existence and long-time behavior of solutions to the multidimensional compressible isentropic Euler equations with damping were studied by many researchers (cf. [15, 24–35] and the references cited therein). Recently, the free boundary problem of the Euler equations with damping was considered (cf. [36–38]).

To the best of our knowledge, there are few results on the three-dimensional full compressible damped Euler equations (1). We first notice that the system (1) can be equivalently reduced to the  $(\rho, u, \theta)$ -system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + u \cdot \nabla u + R\theta \frac{\nabla \rho}{\rho} + R\nabla \theta = -\alpha u, \\ c_v \theta_t + c_v u \cdot \nabla \theta + R\theta \operatorname{div} u = \kappa \frac{\Delta \theta}{\rho}, \end{cases} \quad (3)$$

or the  $(P, u, S)$ -system

$$\begin{cases} P_t + u \cdot \nabla P + \frac{c_v + R}{c_v} P \operatorname{div} u = \frac{R\kappa}{c_v} \Delta \theta, \\ u_t + u \cdot \nabla u + \frac{\nabla P}{\rho} = -\alpha u, \\ S_t + u \cdot \nabla S = R\kappa \frac{\Delta \theta}{P}, \end{cases} \quad (4)$$

where  $\rho$  and  $\theta$  are given by

$$\rho = A^{-(c_v/(c_v+R))} P^{c_v/(c_v+R)} e^{-S/(c_v+R)}, \quad \theta = R^{-1} A^{c_v/(c_v+R)} P^{R/(c_v+R)} e^{S/(c_v+R)}. \quad (5)$$

When  $\kappa > 0$ , Chen et al. [39] considered the  $(\rho, u, \theta)$ -system (3) and then used Fourier analysis methods together with energy methods to prove the global existence and time-decay rates of small smooth solutions. For the case of  $\kappa = 0$ , the temperature equation in (3) has no dissipation, and thus, the method used in [39] is not applicable. To overcome the difficulties arising from the nondissipation of  $\theta$ , the researchers in [40, 41] studied the  $(P, u, S)$ -system (4) with  $\kappa = 0$  and thus proved the similar results as the case of  $\kappa > 0$ . An important observation is that the linear parts of  $(P, u)$  and  $S$  are decoupled in the linearized  $(P, u, S)$ -system, which helps to derive the desired estimates as done in [40, 41]. With regard to the corresponding initial-boundary value problem for  $\kappa = 0$  in a bounded domain, Zhang and Wu [42] and Wu [43] independently obtained the global existence and the exponential stability of small smooth solutions.

In the present paper, we shall choose the  $(\rho, u, \theta)$ -system and prove the global existence and uniqueness of the smooth solution to the Cauchy problem (1) near a constant equilibrium state  $(1, 0, 1)$  for the initial data with various regularities. At the same time, we will use a pure energy method developed in [29, 44] to derive the optimal time-decay rates

of solutions as well as their spatial derivatives of any order. Compared with the Fourier analysis method used in [39], the pure energy method can be used to obtain the optimal time-decay rates under the weak regularity assumptions, which can be seen from  $L^p \subset \dot{H}^{-s}$  ( $1 < p \leq 2, 0 \leq s < 3/2$ ) or  $L^p \subset \dot{B}_{2,\infty}^{-s}$  ( $1 \leq p < 2, 0 < s \leq 3/2$ ). As a byproduct, we give the optimal  $L^p$ - $L^q$ -type decay rates of solutions (see Corollary 3).

*Notation.* Throughout this paper,  $\nabla^k$  with an integer  $k \geq 0$  represents the usual any spatial derivatives of order  $k$ . When  $k < 0$  or  $k$  is not a positive integer,  $\nabla^k$  means  $\Lambda^k$  defined by  $\Lambda^k f := \mathcal{F}^{-1}(|\xi|^k \mathcal{F}f)$ , where  $\mathcal{F}$  is the usual Fourier transform operator and  $\mathcal{F}^{-1}$  is its inverse. We denote by  $L^p(\mathbb{R}^3)$  ( $1 \leq p \leq \infty$ ) the usual Lebesgue spaces with the norm  $\|\cdot\|_{L^p}$ . For simplicity, we write  $\|\cdot\| = \|\cdot\|_{L^2}$ . We use  $H^N(\mathbb{R}^3)$  for some integer  $N \geq 0$  to denote the usual Sobolev spaces with the norm  $\|\cdot\|_{H^N}$ . We use  $\dot{H}^s(\mathbb{R}^3)$  ( $s \in \mathbb{R}$ ) to denote the homogeneous Sobolev spaces with the norm  $\|\cdot\|_{\dot{H}^s}$  defined by  $\|f\|_{\dot{H}^s} := \|\nabla^s f\|$ . It is clear for  $H^0 = \dot{H}^0 = L^2$ .

We review the homogeneous Besov spaces. Let  $\phi \in C_0^\infty(\mathbb{R}^3)$  satisfy  $\phi(\xi) = 1$  if  $|\xi| \leq 1$  and  $\phi(\xi) = 0$  if  $|\xi| \geq 2$ . Define  $\varphi(\xi) := \phi(\xi) - \phi(2\xi)$  and  $\varphi_j(\xi) := \varphi(2^{-j}\xi)$  for  $j \in \mathbb{Z}$ . Then,  $\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1$  if  $\xi \neq 0$ . Define  $\dot{\Delta}_j f := \mathcal{F}^{-1}(\varphi_j) * f$ . For  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ , we denote by  $\dot{B}_{p,\infty}^s(\mathbb{R}^3)$  the homogeneous Besov spaces with the norm  $\|\cdot\|_{\dot{B}_{p,\infty}^s}$  defined by  $\|f\|_{\dot{B}_{p,\infty}^s} := \sup_{j \in \mathbb{Z}} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}$ .

We employ the notation  $A \leq B$  to mean that  $A \leq CB$  for a generic positive constant  $C$ . We denote  $A \sim B$  if  $A \leq B$  and  $B \leq A$ . We use  $C_0$  to denote a positive constant depending additionally on the initial data. For simplicity, we write  $\|(A, B)\|_X := \|A\|_X + \|B\|_X$  and  $\int f := \int_{\mathbb{R}^3} f dx$ . The notation  $C^k(0, T; X)$  ( $k \geq 0$ ) denotes the space of  $X$ -valued  $k$ -times continuously differentiable functions on  $[0, T]$ .

The main results in this paper can be stated as follows.

**Theorem 1.** *Let  $N \geq 3$  be an integer. Assume that  $(\rho_0 - 1, u_0, \theta_0 - 1) \in H^N$  satisfying*

$$\|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^3} < \delta_0, \quad \text{if } N = 3, \quad (6)$$

or

$$\|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^4} < \delta_0, \quad \text{if } N \geq 4, \quad (7)$$

for some small constant  $\delta_0 > 0$ . Then, the Cauchy problem (1) admits a unique global solution  $(\rho, u, \theta)(t)$  such that for all  $t \geq 0$  and  $3 \leq \ell \leq N$ ,

$$\begin{aligned} \|(\rho - 1, u, \theta - 1)(t)\|_{H^\ell} + \left( \int_0^t (\|\nabla \rho(\tau)\|_{H^{\ell-1}}^2 + \|(u, \nabla \theta)(\tau)\|_{H^\ell}^2) d\tau \right)^{1/2} \\ \leq C \|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^\ell}. \end{aligned} \quad (8)$$

**Theorem 2.** Under the assumptions of Theorem 1, if further  $(\rho_0 - 1, u_0, \theta_0 - 1) \in \dot{H}^{-s}$  for some  $s \in [0, 3/2]$ , or  $(\rho_0 - 1, u_0, \theta_0 - 1) \in \dot{B}_{2,\infty}^{-s}$  for some  $s \in (0, 3/2]$ , then for all  $t \geq 0$ ,

$$\begin{cases} \left\| \nabla^l(\rho - 1, \theta - 1)(t) \right\| \leq C_0(1+t)^{-((l+s)/2)}, & 0 \leq l \leq N-1, \\ \left\| \nabla^N(\rho - 1, \theta - 1)(t) \right\| \leq C_0(1+t)^{-((N-1+s)/2)}, \end{cases} \quad (9)$$

and

$$\begin{cases} \left\| \nabla^l u(t) \right\| \leq C_0(1+t)^{-((l+1+s)/2)}, & 0 \leq l \leq N-2, \\ \left\| (\nabla^{N-1} u, \nabla^N u)(t) \right\| \leq C_0(1+t)^{-((N-1+s)/2)}. \end{cases} \quad (10)$$

By Lemma 9 and Lemmas 13 and 14, we easily obtain the following  $L^p-L^q$ -type decay rates.

**Corollary 3.** Under the assumptions of Theorem 2, if  $(\rho_0 - 1, u_0, \theta_0 - 1) \in L^p$  for some  $p \in [1, 2]$ , then for  $2 \leq q \leq \infty$ ,

$$\begin{cases} \left\| \nabla^l(\rho - 1, \theta - 1)(t) \right\|_{L^q} \leq C_0(1+t)^{-(3/2)((1/p)-(1/q))-(l/2)}, & 0 \leq l \leq N-3, \\ \left\| \nabla^l u(t) \right\|_{L^q} \leq C_0(1+t)^{-(3/2)((1/p)-(1/q))-(l+1/2)}, & 0 \leq l \leq N-4, \\ \left\| \nabla^{N-3} u(t) \right\|_{L^q} \leq C_0(1+t)^{-(3/2)((1/p)-(1/q))-(2N-5)/4-(1/2q)}. \end{cases} \quad (11)$$

We give some remarks for Theorems 1 and 2 and Corollary 3.

*Remark 4.* From Theorem 1, when  $N \geq 5$ , we only require that the  $H^4$  norms of the initial density, velocity, and temperature be small, while the higher-order Sobolev norms can be arbitrarily large.

*Remark 5.* We claim that the decay rates except the velocity  $u$  in Theorem 2 and Corollary 3 are optimal in the sense that they are consistent with those in the linearized case.

*Remark 6.* By Corollary 3, we prove the optimal  $L^p-L^q$ -type time-decay rates without the smallness assumption on the  $L^p(1 \leq p < 2)$  norm of the initial data.

*Remark 7.* Compared with the decay results of the full compressible Navier-Stokes equations [44, 45]

$$\|(\rho - 1, u, \theta - 1)\| \leq C_0(1+t)^{-(3/4)}, \quad (12)$$

the density and temperature of the full compressible damped Euler equations have the same decay rates (see (9) with  $l=0$  and  $s=3/2$ ); however, the decay of the  $L^2$  norm of the velocity is improved to  $(1+t)^{-(5/4)}$  (see (10) with  $l=0$  and  $s=3/2$ ) due to the damping effect.

*Remark 8.* With regard to the initial-boundary value problem of the three-dimensional full compressible damped Euler

equations (1), the case of  $\kappa=0$  was solved in [42, 43], and the corresponding  $(P, u, S)$ -system was adopted. For the case of  $\kappa > 0$ , we believe that it is more convenient to deal with the  $(\rho, u, \theta)$ -system, which is a forthcoming work.

The arrangement of this paper is as follows. In Section 2, we list some useful lemmas which will be frequently used. In Section 3, we establish some refined energy estimates (see Lemmas 17–19) which help us to derive important energy estimates with the minimum derivatives counts (see Lemma 20). Then, we prove the global solution (Theorem 1) and the time-decay rates (Theorem 2) in Sections 4 and 5, respectively.

## 2. Preliminaries

In this section, we will give some lemmas which are often used in the later sections. We first recall the Gagliardo-Nirenberg-Sobolev inequality.

**Lemma 9.** Let  $0 \leq m, \alpha \leq l$ , and  $2 \leq p \leq \infty$ . Then, we have

$$\|\nabla^\alpha f\|_{L^p} \leq \|\nabla^m f\|^{1-\vartheta} \|\nabla^l f\|^\vartheta, \quad (13)$$

where  $0 \leq \vartheta \leq 1$  and  $\alpha$  satisfy

$$\alpha + 3\left(\frac{1}{2} - \frac{1}{p}\right) = m(1-\vartheta) + l\vartheta. \quad (14)$$

Here, we require that  $0 < \vartheta < 1$ ,  $m \leq \alpha + 1$ , and  $l \geq \alpha + 2$ , when  $p = \infty$ .

*Proof.* (see [46], Theorem, p.125).

We give the commutator and product estimates.

**Lemma 10.** Let  $m$  be an integer. Define the commutator

$$[\nabla^m, f]g := \nabla^m(fg) - f\nabla^m g. \quad (15)$$

Then, we have for  $m \geq 1$ ,

$$\|[\nabla^m, f]g\|_{L^p} \leq \|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \quad (16)$$

and for  $m \geq 0$ ,

$$\|\nabla^m(fg)\|_{L^p} \leq \|f\|_{L^{p_1}} \|\nabla^m g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \quad (17)$$

where  $p, p_1, p_2, p_3, p_4 \in [1, \infty]$  and  $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$ .

*Proof.* (refer to [47], Lemma 3.1, or [48], Lemma A.4).

The following lemma gives the convenient  $L^p$  estimates for well-prepared functions.

**Lemma 11.** Assume that  $\|q\|_{L^\infty} \leq 1$  and  $\|\Theta\|_{L^\infty} \leq 1$ . Let  $g(q, \Theta)$  be a smooth function of  $q, \Theta$  with bounded derivatives of any order, then for any integer  $k \geq 1$  and  $2 \leq p \leq \infty$ ,

$$\left\| \nabla^k (g(\mathbf{q}, \Theta)) \right\|_{L^p} \leq \left\| \nabla^k \mathbf{q} \right\|_{L^p} + \left\| \nabla^k \Theta \right\|_{L^p}. \quad (18)$$

*Proof.* (see [49], Lemma A.2).

As a byproduct of Lemma 11, we immediately have the following.

**Corollary 12.** *Assume that  $\|\mathbf{q}\|_{L^\infty} \leq 1$ . Let  $g(\mathbf{q})$  be a smooth function of  $\mathbf{q}$  with bounded derivatives of any order, then for any integer  $k \geq 1$  and  $2 \leq p \leq \infty$ ,*

$$\left\| \nabla^k (g(\mathbf{q})) \right\|_{L^p} \leq \left\| \nabla^k \mathbf{q} \right\|_{L^p}. \quad (19)$$

Finally, we list some useful estimates or interpolation inequalities involving negative Sobolev or Besov spaces.

**Lemma 13.** *Let  $1 < p \leq 2$  and  $1/2 + s/3 = 1/p$ . Then,  $0 \leq s < 3/2$  and*

$$\|f\|_{\dot{H}^{-s}} \leq \|f\|_{L^p}. \quad (20)$$

*Proof.* It follows from the Hardy-Littlewood-Sobolev theorem (cf. [50], Theorem 1, p.119).

**Lemma 14.** *Let  $1 \leq p < 2$  and  $1/2 + s/3 = 1/p$ . Then,  $0 < s \leq 3/2$  and*

$$\|f\|_{\dot{B}_{2,\infty}^{-s}} \leq \|f\|_{L^p}. \quad (21)$$

*Proof.* (see [51], Lemma 4.1).

**Lemma 15.** *Let  $s \geq 0$  and  $l \geq 0$ . Then,*

$$\left\| \nabla^l f \right\| \leq \left\| \nabla^{l+1} f \right\|^{1-\vartheta} \|f\|_{\dot{H}^{-s}}^\vartheta, \quad \vartheta = \frac{1}{l+s+1}. \quad (22)$$

*Proof.* (see [44], Lemma A.4).

**Lemma 16.** *Let  $s > 0$  and  $l \geq 0$ . Then,*

$$\left\| \nabla^l f \right\| \leq \left\| \nabla^{l+1} f \right\|^{1-\vartheta} \|f\|_{\dot{B}_{2,\infty}^{-s}}^\vartheta, \quad \vartheta = \frac{1}{l+s+1}. \quad (23)$$

*Proof.* We refer to [51], Lemma 4.2, by noting that  $\dot{B}_{2,p}^{-s} \subset \dot{B}_{2,q}^{-s}$  for  $p \leq q$ .

### 3. Energy Estimates

By a simple calculation, the Cauchy problem (1) becomes

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + u \cdot \nabla u + R\theta \frac{\nabla \rho}{\rho} + R\nabla \theta = -\alpha u, \\ c_v \theta_t + c_v u \cdot \nabla \theta + R\theta \operatorname{div} u = \kappa \frac{\Delta \theta}{\rho}, \\ (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0). \end{cases} \quad (24)$$

Without loss of generality, we assume  $R = c_v = \alpha = \kappa = 1$  and choose the constant equilibrium state  $(1, 0, 1)$ . Define the perturbations

$$\mathbf{q} = \rho - 1, \quad u = u - 0, \quad \Theta = \theta - 1. \quad (25)$$

Then, problem (24) is reformulated as

$$\begin{cases} \mathbf{q}_t = -u \cdot \nabla \mathbf{q} - (1 + \mathbf{q}) \operatorname{div} u, \\ u_t + u \cdot \nabla u + \nabla \Theta = -u \cdot \nabla u - \frac{1 + \Theta}{1 + \mathbf{q}} \nabla \mathbf{q}, \\ \Theta_t - \Delta \Theta = -u \cdot \nabla \Theta - (1 + \Theta) \operatorname{div} u - \frac{\mathbf{q}}{1 + \mathbf{q}} \Delta \Theta, \\ (\mathbf{q}, u, \Theta)|_{t=0} = (\mathbf{q}_0, u_0, \Theta_0) := (\rho_0 - 1, u_0, \theta_0 - 1). \end{cases} \quad (26)$$

We will derive the a priori estimates for the problem (26) by assuming that for sufficiently small  $\delta > 0$  and some  $T > 0$ ,

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^N} < \delta, \quad (27)$$

where  $N = 3$  or  $4$ . By Sobolev's inequality, (27) implies

$$\frac{1}{2} \leq 1 + \mathbf{q} \leq \frac{3}{2}, \quad \frac{1}{2} \leq 1 + \Theta \leq \frac{3}{2}. \quad (28)$$

First, we derive the energy estimates for  $(\mathbf{q}, u, \Theta)$  up to order  $N - 1$ , which contain the dissipation estimates for  $u$  and  $\Theta$  up to order  $N - 1$  and  $N$ , respectively.

**Lemma 17.** *Let  $N \geq 3$  and  $\delta \ll 1$ . If  $\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^3} < \delta$ , then for  $0 \leq k \leq N - 1$ ,*

$$\frac{d}{dt} \left\| \nabla^k (\mathbf{q}, u, \Theta) \right\|^2 + \left\| \nabla^k u \right\|^2 + \left\| \nabla^{k+1} \Theta \right\|^2 \leq \delta \left\| \nabla^{k+1} \mathbf{q} \right\|^2. \quad (29)$$

*Proof.* It is trivial for  $k = 0$ . Next, we will prove (29) for  $1 \leq k \leq N - 1$ . Applying  $\nabla^k$  to (26)<sub>1</sub>, (26)<sub>2</sub>, and (26)<sub>3</sub> and

multiplying the resulting identities by  $\nabla^k \mathbf{q}$ ,  $\nabla^k u$ ,  $\nabla^k \Theta$ , respectively; summing them up; and then integrating over  $\mathbb{R}^3$  by parts, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| \nabla^k (\mathbf{q}, u, \Theta) \right\|^2 + \left\| \nabla^k u \right\|^2 + \left\| \nabla^{k+1} \Theta \right\|^2 \\ &= - \int \nabla^k \operatorname{div} (\mathbf{q}u) \cdot \nabla^k \mathbf{q} - \int \nabla^k \left( u \cdot \nabla u + \frac{\Theta - \mathbf{q}}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \cdot \nabla^k u \\ & \quad - \int \nabla^k \operatorname{div} (\Theta u) \cdot \nabla^k \Theta - \int \nabla^k \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \Delta \Theta \right) \cdot \nabla^k \Theta \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (30)$$

Now, we estimate the terms  $I_1$ – $I_4$ . For the term  $I_1$ , by integrating by parts; Hölder's, Sobolev's, and Cauchy's inequalities; and (17) of Lemma 10, we obtain

$$\begin{aligned} I_1 &= \int \nabla^{k-1} \operatorname{div} (\mathbf{q}u) \cdot \nabla^{k-1} \Delta \mathbf{q} \\ &\leq \left\| \nabla^k (\mathbf{q}u) \right\| \left\| \nabla^{k+1} \mathbf{q} \right\| \\ &\leq \left( \left\| \nabla^k \mathbf{q} \right\|_{L^6} \|u\|_{L^3} + \|\mathbf{q}\|_{L^\infty} \left\| \nabla^k u \right\| \right) \left\| \nabla^{k+1} \mathbf{q} \right\| \\ &\leq \delta \left( \left\| \nabla^{k+1} \mathbf{q} \right\|^2 + \left\| \nabla^k u \right\|^2 \right). \end{aligned} \quad (31)$$

For the term  $I_2$ , by integrating by parts; Hölder's, Sobolev's, and Cauchy's inequalities; Lemma 10; and Corollary 12, we obtain

$$\begin{aligned} \int \nabla^k (u \cdot \nabla u) \cdot \nabla^k u &= \int \left( u \cdot \nabla \nabla^k u \cdot \nabla^k u + [\nabla^k, u] \cdot \nabla u \cdot \nabla^k u \right) \\ &= -\frac{1}{2} \int \operatorname{div} u \left| \nabla^k u \right|^2 + \int [\nabla^k, u] \cdot \nabla u \cdot \nabla^k u \\ &\leq \|\operatorname{div} u\|_{L^\infty} \left\| \nabla^k u \right\|^2 + \left\| [\nabla^k, u] \cdot \nabla u \right\| \left\| \nabla^k u \right\| \\ &\leq \|u\|_{H^3} \left\| \nabla^k u \right\|^2 + \|\nabla u\|_{L^\infty} \left\| \nabla^k u \right\|^2 \\ &\leq \delta \left\| \nabla^k u \right\|^2 \end{aligned} \quad (32)$$

$$\begin{aligned} \int \nabla^k \left( \frac{\Theta - \mathbf{q}}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \cdot \nabla^k u &\leq \left\| \nabla^k \left( \frac{\Theta - \mathbf{q}}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \right\| \left\| \nabla^k u \right\| \\ &\leq \left( \left\| \nabla^k \left( \frac{\nabla \mathbf{q}}{1 + \mathbf{q}} \Theta \right) \right\| + \left\| \nabla^k \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \right\| \right) \left\| \nabla^k u \right\| \\ &\leq \left( \left\| \nabla^k \left( \frac{\nabla \mathbf{q}}{1 + \mathbf{q}} \right) \right\| \|\Theta\|_{L^\infty} + \left\| \frac{\nabla \mathbf{q}}{1 + \mathbf{q}} \right\|_{L^3} \left\| \nabla^k \Theta \right\|_{L^6} \right) \left\| \nabla^k u \right\| \\ & \quad + \left( \left\| \nabla^k \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \right) \right\|_{L^6} \|\nabla \mathbf{q}\|_{L^3} + \left\| \frac{\mathbf{q}}{1 + \mathbf{q}} \right\|_{L^\infty} \left\| \nabla^{k+1} \mathbf{q} \right\| \right) \left\| \nabla^k u \right\| \\ &\leq \delta \left( \left\| \nabla^{k+1} (\mathbf{q}, \Theta) \right\|^2 + \left\| \nabla^k u \right\|^2 \right). \end{aligned} \quad (33)$$

In light of (32) and (33), we have

$$I_2 \leq \delta \left( \left\| \nabla^{k+1} (\mathbf{q}, \Theta) \right\|^2 + \left\| \nabla^k u \right\|^2 \right). \quad (34)$$

As with the term  $I_1$ , we obtain

$$I_3 \leq \delta \left( \left\| \nabla^{k+1} \Theta \right\|^2 + \left\| \nabla^k u \right\|^2 \right). \quad (35)$$

For the term  $I_4$ , by integrating by parts; Hölder's, Sobolev's, and Young's inequalities; Lemmas 9 and 10; and Corollary 12, we obtain

$$\begin{aligned} I_4 &= \int \nabla^{k-1} \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \Delta \Theta \right) \cdot \nabla^{k-1} \Delta \Theta \leq \left\| \nabla^{k-1} \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \Delta \Theta \right) \right\| \left\| \nabla^{k+1} \Theta \right\| \\ &\leq \left( \left\| \nabla^{k-1} \left( \frac{\mathbf{q}}{1 + \mathbf{q}} \right) \right\|_{L^6} \|\Delta \Theta\|_{L^3} + \left\| \frac{\mathbf{q}}{1 + \mathbf{q}} \right\|_{L^\infty} \left\| \nabla^{k+1} \Theta \right\| \right) \left\| \nabla^{k+1} \Theta \right\| \\ &\leq \left( \left\| \nabla^k \mathbf{q} \right\| \left\| \nabla^2 \Theta \right\|_{L^3} + \|\mathbf{q}\|_{H^2} \left\| \nabla^{k+1} \Theta \right\| \right) \left\| \nabla^{k+1} \Theta \right\| \\ &\leq \left\| \nabla^{k+1} \mathbf{q} \right\|^{k/(k+1)} \|\mathbf{q}\|^{1/(k+1)} \left\| \nabla^{k+1} \mathbf{q} \right\|^{1/(k+1)} \left\| \nabla^{3(k+1)/2k} \Theta \right\|^{k/(k+1)} \left\| \nabla^{k+1} \Theta \right\| + \delta \left\| \nabla^{k+1} \Theta \right\|^2 \\ &\leq \|(\mathbf{q}, \Theta)\|_{H^3} \left( \left\| \nabla^{k+1} \mathbf{q} \right\| + \left\| \nabla^{k+1} \Theta \right\| \right) \left\| \nabla^{k+1} \Theta \right\| + \delta \left\| \nabla^{k+1} \Theta \right\|^2 \\ &\leq \delta \left\| \nabla^{k+1} (\mathbf{q}, \Theta) \right\|^2. \end{aligned} \quad (36)$$

Plugging the estimates for  $I_1$ – $I_4$  into (30), we deduce (29).

Next, we derive the  $N$ -th-order energy estimates for  $(\mathbf{q}, u, \Theta)$ , which contain the dissipation estimates for  $u$  and  $\Theta$  of order  $N$  and  $N + 1$ , respectively.

**Lemma 18.** *Let  $N \geq 3$ . Then, we have*

$$\frac{d}{dt} \int \mathfrak{E}^N(t) + \|\nabla^N u\|^2 + \|\nabla^{N+1} \Theta\|^2 \leq \delta \|\nabla^N (\mathbf{q}, \Theta)\|^2, \quad (37)$$

under the assumption of

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^3} < \delta \ll 1, \quad \text{if } N = 3, \quad (38)$$

or

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^4} < \delta \ll 1, \quad \text{if } N \geq 4. \quad (39)$$

Here,

$$\begin{aligned} \mathfrak{E}^N(t) &:= \frac{1 + \Theta}{1 + \mathbf{q}} |\nabla^N \mathbf{q}|^2 + (1 + \mathbf{q}) |\nabla^{N-1} \operatorname{div} u|^2 \\ & \quad + |\nabla^{N-1} \operatorname{curl} u|^2 + \frac{1 + \mathbf{q}}{1 + \Theta} |\nabla^N \Theta|^2. \end{aligned} \quad (40)$$

*Proof.* We first prove (ii). For equations (26)<sub>1</sub>–(26)<sub>3</sub>, computing

$$\begin{aligned} & \int \left[ \frac{1 + \Theta}{1 + \mathbf{q}} \nabla^N \mathbf{q} \cdot \nabla^N (26)_1 + (1 + \mathbf{q}) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \operatorname{div} (26)_2 \right. \\ & \quad \left. + \frac{1 + \mathbf{q}}{1 + \Theta} \nabla^N \Theta \cdot \nabla^N (26)_3 \right] \end{aligned} \quad (41)$$

and integrating by parts, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \left[ \frac{1+\Theta}{1+\varrho} |\nabla^N \varrho|^2 + (1+\varrho) |\nabla^{N-1} \operatorname{div} u|^2 + \frac{1+\varrho}{1+\Theta} |\nabla^N \Theta|^2 \right] \\
& + \int \left[ (1+\varrho) |\nabla^{N-1} \operatorname{div} u|^2 + \frac{1+\varrho}{1+\Theta} |\nabla^{N+1} \Theta|^2 \right] \\
& = \frac{1}{2} \int \left[ \partial_t \left( \frac{1+\Theta}{1+\varrho} \right) |\nabla^N \varrho|^2 + \partial_t \varrho |\nabla^{N-1} \operatorname{div} u|^2 + \partial_t \left( \frac{1+\varrho}{1+\Theta} \right) |\nabla^N \Theta|^2 \right] \\
& - \int \left\{ \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot \nabla^N [(1+\varrho) \operatorname{div} u] \right. \\
& + (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \operatorname{div} \left( \frac{1+\Theta}{1+\varrho} \nabla \varrho \right) \left. \right\} \\
& - \int \left\{ (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \Delta \Theta + \frac{1+\varrho}{1+\Theta} \nabla^N \Theta \cdot \nabla^N [(1+\Theta) \operatorname{div} u] \right\} \\
& - \int \nabla \left( \frac{1+\varrho}{1+\Theta} \right) \cdot \nabla^N \Theta \cdot \nabla^{N+1} \Theta - \int \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot \nabla^N (u \cdot \nabla \varrho) \\
& - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \operatorname{div} (u \cdot \nabla u) - \int \frac{1+\varrho}{1+\Theta} \nabla^N \Theta \cdot \nabla^N (u \cdot \nabla \Theta) \\
& - \int \frac{1+\varrho}{1+\Theta} \nabla^N \Theta \cdot \nabla^N \left( \frac{\varrho}{1+\varrho} \Delta \Theta \right) \\
& =: \sum_{i=1}^8 J_i.
\end{aligned} \tag{42}$$

Now, we estimate the terms  $J_1$ – $J_8$ . By (26)<sub>1</sub>, (26)<sub>3</sub>, (28), and Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}
J_1 & \leq (\|\Theta_t\|_{L^\infty} + \|\varrho_t\|_{L^\infty}) \|\nabla^N(\varrho, u, \Theta)\|^2 \\
& \leq (\|(\varrho, u)\|_{H^3} + \|\Theta\|_{H^4}) \|\nabla^N(\varrho, u, \Theta)\|^2 \\
& \leq \delta \|\nabla^N(\varrho, u, \Theta)\|^2.
\end{aligned} \tag{43}$$

By the commutator notation (15), the commutator estimate (16), integrating by parts, and Lemma 11, we have

$$\begin{aligned}
J_2 & = - \int \left[ (1+\Theta) \nabla^N \varrho \cdot \nabla^N \operatorname{div} u + (1+\Theta) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \Delta \varrho \right] \\
& - \int \left\{ \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot [\nabla^N, 1+\varrho] \operatorname{div} u + (1+\varrho) \nabla^{N-1} \operatorname{div} u \right. \\
& \cdot \left. \left[ \nabla^{N-1} \operatorname{div}, \frac{1+\Theta}{1+\varrho} \right] \nabla \varrho \right\} \\
& \leq \int |\nabla \Theta| |\nabla^N \varrho| |\nabla^N u| + \int |\nabla^N \varrho| |\nabla^N, 1 \\
& + \varrho] \nabla u| + \int |\nabla^N u| \left| \left[ \nabla^N, \frac{1+\Theta}{1+\varrho} \right] \nabla \varrho \right| \\
& \leq \|\nabla \Theta\|_{L^\infty} \|\nabla^N \varrho\| \|\nabla^N u\| + \|\nabla^N \varrho\| \|\nabla^N, 1 \\
& + \varrho] \nabla u\| + \|\nabla^N u\| \left\| \left[ \nabla^N, \frac{1+\Theta}{1+\varrho} \right] \nabla \varrho \right\| \\
& \leq \|\nabla \Theta\|_{L^\infty} \|\nabla^N u\| \|\nabla^N \varrho\| + \|\nabla^N \varrho\| (\|\nabla \varrho\|_{L^\infty} \|\nabla^N u\| \\
& + \|\nabla^N \varrho\| \|\nabla u\|_{L^\infty}) + \|\nabla^N u\| \left( \left\| \nabla \left( \frac{1+\Theta}{1+\varrho} \right) \right\|_{L^\infty} \|\nabla^N \varrho\| \right. \\
& + \left. \left\| \nabla^N \left( \frac{1+\Theta}{1+\varrho} \right) \right\| \|\nabla \varrho\|_{L^\infty} \right) \\
& \leq \delta \|\nabla^N(\varrho, u, \Theta)\|^2,
\end{aligned}$$

$$\begin{aligned}
J_3 & = - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \Delta \Theta + (1+\varrho) \nabla^N \Theta \cdot \nabla^N \operatorname{div} u \\
& - \int \frac{1+\varrho}{1+\Theta} \nabla^N \Theta \cdot [\nabla^N, 1+\Theta] \operatorname{div} u \\
& \leq \int |\nabla \varrho| |\nabla^N \Theta| |\nabla^{N-1} \operatorname{div} u| \\
& + \int |\nabla^N \Theta| \|\nabla^N, 1+\Theta\| \|\nabla u\| \\
& \leq \|\nabla \varrho\|_{L^\infty} \|\nabla^N \Theta\| \|\nabla^{N-1} \operatorname{div} u\| \\
& + \|\nabla^N \Theta\| (\|\nabla \Theta\|_{L^\infty} \|\nabla^N u\| + \|\nabla^N \Theta\| \|\nabla u\|_{L^\infty}) \\
& \leq \delta \|\nabla^N(u, \Theta)\|^2.
\end{aligned} \tag{44}$$

By Hölder's, Sobolev's, and Cauchy's inequalities and Lemma 11, we obtain

$$J_4 \leq \left\| \nabla \left( \frac{1+\varrho}{1+\Theta} \right) \right\|_{L^3} \|\nabla^N \Theta\|_{L^6} \|\nabla^{N+1} \Theta\| \leq \delta \|\nabla^{N+1} \Theta\|^2. \tag{45}$$

By Lemma 10, integrating by parts, and Hölder's, Sobolev's, and Cauchy's inequalities, we have

$$\begin{aligned}
J_5 & = - \int \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot \nabla^N (u \cdot \nabla \varrho) \\
& = - \frac{1}{2} \int \frac{1+\Theta}{1+\varrho} u \cdot \nabla |\nabla^N \varrho|^2 - \int \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot [\nabla^N, u] \cdot \nabla \varrho \\
& = \frac{1}{2} \int \operatorname{div} \left( \frac{1+\Theta}{1+\varrho} u \right) |\nabla^N \varrho|^2 - \int \frac{1+\Theta}{1+\varrho} \nabla^N \varrho \cdot [\nabla^N, u] \cdot \nabla \varrho \\
& \leq \delta \|\nabla^N \varrho\|^2 + \|\nabla^N \varrho\| \|\nabla^N, u\| \|\nabla \varrho\| \\
& \leq \delta \|\nabla^N \varrho\|^2 + \|\nabla^N \varrho\| (\|\nabla u\|_{L^\infty} \|\nabla^N \varrho\| + \|\nabla^N u\| \|\nabla \varrho\|_{L^\infty}) \\
& \leq \delta \|\nabla^N(\varrho, u)\|^2.
\end{aligned} \tag{46}$$

Note that

$$\operatorname{div} (u \cdot \nabla u) = \nabla u : (\nabla u)^T + u \cdot \nabla \operatorname{div} u, \tag{47}$$

where the double dots : mean that  $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^3 a_{ij} b_{ij}$  for two  $3 \times 3$  matrices  $\mathbb{A} = (a_{ij})$  and  $\mathbb{B} = (b_{ij})$ . By Hölder's, Sobolev's, and Cauchy's inequalities and Lemma 10, we estimate

$$\begin{aligned}
J_6 & = - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \operatorname{div} (u \cdot \nabla u) \\
& = - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \left[ \nabla u : (\nabla u)^T + u \cdot \nabla \operatorname{div} u \right] \\
& = - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \left[ \nabla u : (\nabla u)^T \right] \\
& - \int (1+\varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} (u \cdot \nabla \operatorname{div} u)
\end{aligned}$$

$$\begin{aligned}
&= - \int (1 + \varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \left[ \nabla u : (\nabla u)^T \right] \\
&\quad - \frac{1}{2} \int (1 + \varrho) u \cdot \nabla \left| \nabla^{N-1} \operatorname{div} u \right|^2 \\
&\quad - \int (1 + \varrho) \nabla^{N-1} \operatorname{div} u \cdot \left[ \nabla^{N-1}, u \right] \cdot \nabla \operatorname{div} u \\
&= - \int (1 + \varrho) \nabla^{N-1} \operatorname{div} u \cdot \nabla^{N-1} \left[ \nabla u : (\nabla u)^T \right] \\
&\quad + \frac{1}{2} \int \operatorname{div} \left[ (1 + \varrho) u \right] \left| \nabla^{N-1} \operatorname{div} u \right|^2 \\
&\quad - \int (1 + \varrho) \nabla^{N-1} \operatorname{div} u \cdot \left[ \nabla^{N-1}, u \right] \cdot \nabla \operatorname{div} u \\
&\leq \left\| \nabla^{N-1} \operatorname{div} u \right\| \left\| \nabla^{N-1} \left[ \nabla u : (\nabla u)^T \right] \right\| \\
&\quad + \left\| \operatorname{div} \left[ (1 + \varrho) u \right] \right\|_{L^\infty} \left\| \nabla^{N-1} \operatorname{div} u \right\|^2 \\
&\quad + \left\| \nabla^{N-1} \operatorname{div} u \right\| \left\| \left[ \nabla^{N-1}, u \right] \cdot \nabla \operatorname{div} u \right\| \\
&\leq \left\| \nabla^N u \right\| \left\| \nabla u \right\|_{L^\infty} \left\| \nabla^N u \right\| + \delta \left\| \nabla^N u \right\|^2 \\
&\quad + \left\| \nabla^N u \right\| \left( \left\| \nabla^{N-1} u \right\|_{L^6} \left\| \nabla \operatorname{div} u \right\|_{L^3} + \left\| \nabla u \right\|_{L^\infty} \left\| \nabla^{N-1} \operatorname{div} u \right\| \right) \\
&\leq \delta \left\| \nabla^N u \right\|^2, \\
J_7 &= - \int \frac{1 + \varrho}{1 + \Theta} \nabla^N \Theta \cdot \nabla^N (u \cdot \nabla \Theta) \\
&\leq \left\| \nabla^N \Theta \right\| \left\| \nabla^N (u \cdot \nabla \Theta) \right\| \\
&\leq \left\| \nabla^N \Theta \right\| \left( \left\| u \right\|_{L^\infty} \left\| \nabla^{N+1} \Theta \right\| + \left\| \nabla^N u \right\| \left\| \nabla \Theta \right\|_{L^\infty} \right) \\
&\leq \delta \left( \left\| \nabla^N (u, \Theta) \right\|^2 + \left\| \nabla^{N+1} \Theta \right\|^2 \right). \tag{48}
\end{aligned}$$

By integrating by parts and by the product estimates (17) of Lemma 10 and Corollary 12, we have

$$\begin{aligned}
J_8 &= - \int \frac{1 + \varrho}{1 + \Theta} \nabla^N \Theta \cdot \nabla^N \left( \frac{\varrho}{1 + \varrho} \Delta \Theta \right) \\
&= \int \operatorname{div} \left( \frac{1 + \varrho}{1 + \Theta} \nabla^N \Theta \right) \cdot \nabla^{N-1} \left( \frac{\varrho}{1 + \varrho} \Delta \Theta \right) \\
&= \int \left[ \nabla \left( \frac{1 + \varrho}{1 + \Theta} \right) \nabla^N \Theta + \frac{1 + \varrho}{1 + \Theta} \nabla^{N+1} \Theta \right] \cdot \nabla^{N-1} \left( \frac{\varrho}{1 + \varrho} \Delta \Theta \right) \\
&\leq \left( \left\| \nabla \left( \frac{1 + \varrho}{1 + \Theta} \right) \right\|_{L^3} \left\| \nabla^N \Theta \right\|_{L^6} + \left\| \nabla^{N+1} \Theta \right\| \right) \left\| \nabla^{N-1} \left( \frac{\varrho}{1 + \varrho} \Delta \Theta \right) \right\| \\
&\leq \left( \left\| \nabla \left( \frac{1 + \varrho}{1 + \Theta} \right) \right\|_{L^3} \left\| \nabla^N \Theta \right\|_{L^6} + \left\| \nabla^{N+1} \Theta \right\| \right) \\
&\quad \cdot \left( \left\| \nabla^{N-1} \left( \frac{\varrho}{1 + \varrho} \right) \right\|_{L^6} \left\| \Delta \Theta \right\|_{L^3} + \left\| \varrho \right\|_{L^\infty} \left\| \nabla^{N+1} \Theta \right\| \right) \\
&\leq \left( \left\| \varrho, \Theta \right\|_{H^3} \left\| \nabla^{N+1} \Theta \right\| + \left\| \nabla^{N+1} \Theta \right\| \right) \left\| \nabla^N \varrho \right\| \left\| \Theta \right\|_{H^3} + \left\| \varrho \right\|_{H^3} \left\| \nabla^{N+1} \Theta \right\| \\
&\leq \delta \left( \left\| \nabla^N \varrho \right\|^2 + \left\| \nabla^{N+1} \Theta \right\|^2 \right). \tag{49}
\end{aligned}$$

Plugging the estimates for  $J_1$ - $J_8$  into (42), by (28), since  $\delta$  is small, we deduce

$$\begin{aligned}
&\frac{d}{dt} \int \left[ \frac{1 + \Theta}{1 + \varrho} \left| \nabla^N \varrho \right|^2 + (1 + \varrho) \left| \nabla^{N-1} \operatorname{div} u \right|^2 + \frac{1 + \varrho}{1 + \Theta} \left| \nabla^N \Theta \right|^2 \right] \\
&\quad + \left\| \nabla^{N-1} \operatorname{div} u \right\|^2 + \left\| \nabla^{N+1} \Theta \right\|^2 \\
&\leq \delta \left\| \nabla^N (\varrho, u, \Theta) \right\|^2. \tag{50}
\end{aligned}$$

Rewrite (26)<sub>2</sub> as

$$u_t + u = -\nabla \varrho - \nabla \Theta - u \cdot \nabla u + \frac{\varrho - \Theta}{1 + \varrho} \nabla \varrho. \tag{51}$$

Applying curl to (51), we obtain

$$(\operatorname{curl} u)_t + \operatorname{curl} u = -\operatorname{curl} (u \cdot \nabla u) + \nabla \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \times \nabla \varrho, \tag{52}$$

where  $\times$  represents the cross product of vectors. Applying  $\nabla^{N-1}$  to (52), multiplying the resulting identity by  $\nabla^{N-1} \operatorname{curl} u$  and integrating over  $\mathbb{R}^3$ , we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \left| \nabla^{N-1} \operatorname{curl} u \right|^2 + \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2 \\
&= - \int \nabla^{N-1} \operatorname{curl} (u \cdot \nabla u) \cdot \nabla^{N-1} \operatorname{curl} u \\
&\quad + \int \nabla^{N-1} \left[ \nabla \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \times \nabla \varrho \right] \cdot \nabla^{N-1} \operatorname{curl} u \\
&:= J_9 + J_{10}. \tag{53}
\end{aligned}$$

By Hölder's, Sobolev's, and Cauchy's inequalities and Lemmas 10 and 11, we estimate

$$\begin{aligned}
J_9 &= - \int \nabla^{N-1} \operatorname{curl} (u \cdot \nabla u) \cdot \nabla^{N-1} \operatorname{curl} u \\
&= - \frac{1}{2} \int u \cdot \nabla \left| \nabla^{N-1} \operatorname{curl} u \right|^2 - \int \left[ \nabla^{N-1} \operatorname{curl}, u \right] \cdot \nabla u \cdot \nabla^{N-1} \operatorname{curl} u \\
&= \frac{1}{2} \int \operatorname{div} u \left| \nabla^{N-1} \operatorname{curl} u \right|^2 - \int \left[ \nabla^{N-1} \operatorname{curl}, u \right] \cdot \nabla u \cdot \nabla^{N-1} \operatorname{curl} u \\
&\leq \delta \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2 + \int \left[ \nabla^N, u \right] \cdot \nabla u \left\| \nabla^N u \right\| \\
&\leq \delta \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2 + \left\| \left[ \nabla^N, u \right] \cdot \nabla u \right\| \left\| \nabla^N u \right\| \\
&\leq \delta \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2 + \left\| \nabla u \right\|_{L^\infty} \left\| \nabla^N u \right\|^2 \leq \delta \left\| \nabla^N u \right\|^2, \\
J_{10} &= \int \nabla^{N-1} \left[ \nabla \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \times \nabla \varrho \right] \cdot \nabla^{N-1} \operatorname{curl} u \\
&\leq \left\| \nabla^{N-1} \left[ \nabla \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \times \nabla \varrho \right] \right\| \left\| \nabla^{N-1} \operatorname{curl} u \right\| \\
&\leq \left( \left\| \nabla^N \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \right\| \left\| \nabla \varrho \right\|_{L^\infty} \right. \\
&\quad \left. + \left\| \nabla \left( \frac{\varrho - \Theta}{1 + \varrho} \right) \right\|_{L^\infty} \left\| \nabla^N \varrho \right\| \right) \left\| \nabla^{N-1} \operatorname{curl} u \right\| \\
&\leq \left( \left\| \nabla^N (\varrho, \Theta) \right\| \left\| \varrho \right\|_{H^3} + \left\| \varrho, \Theta \right\|_{H^3} \left\| \nabla^N \varrho \right\| \right) \left\| \nabla^N u \right\| \\
&\leq \delta \left\| \nabla^N (\varrho, u, \Theta) \right\|^2. \tag{54}
\end{aligned}$$

Plugging the estimates for  $J_9$ - $J_{10}$  into (53), we obtain

$$\frac{d}{dt} \int \left| \nabla^{N-1} \operatorname{curl} u \right|^2 + \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2 \leq \delta \left\| \nabla^N (\mathbf{q}, u, \Theta) \right\|^2. \quad (55)$$

Adding (55) to (50), noting

$$\left\| \nabla^N u \right\|^2 = \left\| \nabla^{N-1} \operatorname{div} u \right\|^2 + \left\| \nabla^{N-1} \operatorname{curl} u \right\|^2, \quad (56)$$

since  $\delta$  is small, we deduce (37).

Now, we prove (i). Note that all the estimates for  $J_2$ - $J_{10}$  in the proof of (ii) also hold under the assumptions of  $N = 3$  and  $\sup_{0 \leq t \leq T} \left\| (\mathbf{q}, u, \Theta)(t) \right\|_{H^3} < \delta$ . Next, we only need to estimate the term  $J_1$  for  $N = 3$  under the condition of  $\sup_{0 \leq t \leq T} \left\| (\mathbf{q}, u, \Theta)(t) \right\|_{H^3} < \delta$ . Note that from (26)<sub>1,2</sub>

$$\begin{aligned} \mathbf{q}_t &= -\mathbf{q} \operatorname{div} u - \operatorname{div} u - u \cdot \nabla \mathbf{q}, \\ \Theta_t &= -u \cdot \nabla \Theta - \operatorname{div} u - \Theta \operatorname{div} u + \frac{1}{1+\mathbf{q}} \Delta \Theta. \end{aligned} \quad (57)$$

For  $N = 3$ , if  $\sup_{0 \leq t \leq T} \left\| (\mathbf{q}, u, \Theta)(t) \right\|_{H^3} < \delta$ , then we can estimate

$$\begin{aligned} J_1 &= \frac{1}{2} \int \left[ \partial_t \left( \frac{1+\Theta}{1+\mathbf{q}} \right) \left| \nabla^3 \mathbf{q} \right|^2 + \partial_t \mathbf{q} \left| \nabla^2 \operatorname{div} u \right|^2 + \partial_t \left( \frac{1+\mathbf{q}}{1+\Theta} \right) \left| \nabla^3 \Theta \right|^2 \right] \\ &\leq \left( \left\| \Theta_t \right\|_{L^\infty} + \left\| \mathbf{q}_t \right\|_{L^\infty} \right) \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 \\ &\leq \left( \left\| \mathbf{q} \right\|_{L^\infty} \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla u \right\|_{L^\infty} + \left\| u \right\|_{L^\infty} \left\| \nabla \mathbf{q} \right\|_{L^\infty} \right) \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 \\ &\quad + \left( \left\| u \right\|_{L^\infty} \left\| \nabla \Theta \right\|_{L^\infty} + \left\| \nabla u \right\|_{L^\infty} \right. \\ &\quad \left. + \left\| \Theta \right\|_{L^\infty} \left\| \nabla u \right\|_{L^\infty} \right) \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 \\ &\quad + \left\| \nabla^2 \Theta \right\|_{L^\infty} \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 \\ &\leq \delta \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 + \left\| \nabla^2 \Theta \right\|_{L^\infty} \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 \\ &\leq \delta \left( \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 + \left\| \nabla^2 \Theta \right\|_{L^\infty}^2 \right) \\ &\leq \delta \left( \left\| \nabla^3 (\mathbf{q}, u, \Theta) \right\|^2 + \left\| \nabla^4 \Theta \right\|^2 \right), \end{aligned} \quad (58)$$

where we have used the interpolation estimate

$$\left\| \nabla^2 \Theta \right\|_{L^\infty} \leq \left\| \nabla^3 \Theta \right\|^{1/2} \left\| \nabla^4 \Theta \right\|^{1/2}. \quad (59)$$

Thus, combining the new estimate (58) with the estimates for  $J_2$ - $J_{10}$  with  $N = 3$ , we can deduce from (42) that (37) holds for  $N = 3$ .

We shall derive the dissipation estimates for  $\mathbf{q}$  up to order  $N$ .

**Lemma 19.** *Let  $N \geq 3$  and  $\delta \ll 1$ . If  $\sup_{0 \leq t \leq T} \left\| (\mathbf{q}, u, \Theta)(t) \right\|_{H^3} < \delta$ , then for  $0 \leq k \leq N - 1$ ,*

$$\frac{d}{dt} \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \left\| \nabla^{k+1} \mathbf{q} \right\|^2 \leq \left\| \nabla^k u \right\|^2 + \left\| \nabla^{k+1} (u, \Theta) \right\|^2. \quad (60)$$

*Proof.* Rewrite (26)<sub>2</sub> as

$$\nabla \mathbf{q} = -u_t - u \cdot \nabla \Theta - u \cdot \nabla u + \frac{\mathbf{q} - \Theta}{1 + \mathbf{q}} \nabla \mathbf{q}. \quad (61)$$

Applying  $\nabla^k$  to (61) and multiplying the resulting identity by  $\nabla \nabla^k \mathbf{q}$  and then integrating over  $\mathbb{R}^3$  by parts and by Hölder's and Cauchy's inequalities, we have

$$\begin{aligned} \left\| \nabla^{k+1} \mathbf{q} \right\|^2 &\leq - \int \partial_t \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \left\| \nabla^k u \right\|^2 \\ &\quad + \left\| \nabla^{k+1} \Theta \right\|^2 + \left\| \nabla^k (u \cdot \nabla u) \right\|^2 + \left\| \nabla^k \left( \frac{\mathbf{q} - \Theta}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \right\|^2. \end{aligned} \quad (62)$$

By (26)<sub>1</sub>, we integrate by parts to obtain

$$\begin{aligned} - \int \partial_t \nabla^k u \cdot \nabla \nabla^k \mathbf{q} &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q}_t \\ &= - \frac{d}{dt} \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \int \nabla^k \operatorname{div} u \\ &\quad \cdot \left[ \nabla^k \operatorname{div} u + \nabla^k \operatorname{div} (\mathbf{q} u) \right] \\ &\leq - \frac{d}{dt} \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \left\| \nabla^{k+1} u \right\|^2 + \left\| \nabla^{k+1} (\mathbf{q} u) \right\|^2 \\ &\leq - \frac{d}{dt} \int \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + \left\| \nabla^{k+1} u \right\|^2 + \delta \left\| \nabla^{k+1} (\mathbf{q}, u) \right\|^2, \end{aligned} \quad (63)$$

where we have used the product estimates (17) of Lemma 10 to estimate

$$\left\| \nabla^{k+1} (\mathbf{q} u) \right\| \leq \left\| \mathbf{q} \right\|_{L^\infty} \left\| \nabla^{k+1} u \right\| + \left\| \nabla^{k+1} \mathbf{q} \right\| \left\| u \right\|_{L^\infty} \leq \delta \left\| \nabla^{k+1} (\mathbf{q}, u) \right\|. \quad (64)$$

By Lemmas 10 and 11, we have

$$\left\| \nabla^k (u \cdot \nabla u) \right\| \leq \left\| u \right\|_{L^\infty} \left\| \nabla^{k+1} u \right\| + \left\| \nabla^k u \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \leq \delta \left\| \nabla^{k+1} u \right\|, \quad (65)$$

$$\begin{aligned} \left\| \nabla^k \left( \frac{\mathbf{q} - \Theta}{1 + \mathbf{q}} \nabla \mathbf{q} \right) \right\| &\leq \left\| (\mathbf{q}, \Theta) \right\|_{L^\infty} \left\| \nabla^{k+1} \mathbf{q} \right\| \\ &\quad + \left\| \nabla^k \left( \frac{\mathbf{q} - \Theta}{1 + \mathbf{q}} \right) \right\|_{L^6} \left\| \nabla \mathbf{q} \right\|_{L^3} \\ &\leq \delta \left\| \nabla^{k+1} (\mathbf{q}, \Theta) \right\|. \end{aligned} \quad (66)$$

Plugging (63)-(66) into (62), we deduce (60).



Finally, we collect all the dissipation estimates for  $(\mathbf{q}, u, \Theta)$  in Lemmas 17–19 to derive the following energy inequality with the minimum derivative counts.

**Lemma 20.** *Let  $N \geq 3$  and  $T > 0$ . Then, there exists an energy functional  $\mathcal{E}_l^N(t)$ , which is equivalent to  $\|\nabla^l(\mathbf{q}, u, \Theta)(t)\|_{H^{N-l}}^2$ , such that for any  $t \in [0, T]$  and  $0 \leq l \leq N-1$ ,*

$$\frac{d}{dt} \mathcal{E}_l^N(t) + \left\| \nabla^{l+1} \mathbf{q}(t) \right\|_{H^{N-l-1}}^2 + \left\| \nabla^l u(t) \right\|_{H^{N-l}}^2 + \left\| \nabla^{l+1} \Theta(t) \right\|_{H^{N-l}}^2 \leq 0, \quad (67)$$

under the assumption of

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^3} < \delta \ll 1, \quad \text{if } N = 3, \quad (68)$$

or

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^4} < \delta \ll 1, \quad \text{if } N \geq 4. \quad (69)$$

*Proof.* Let  $N \geq 3$  and  $0 \leq l \leq N-1$ . Summing up (29) of Lemma 17 from  $k=l$  to  $k=N-1$  and adding the resulting identity to (37) of Lemma 18, since  $\delta$  is small, we obtain

$$\begin{aligned} & \frac{d}{dt} \left[ \left\| \nabla^l(\mathbf{q}, u, \Theta)(t) \right\|_{H^{N-l-1}}^2 + \int \mathfrak{E}^N(t) \right] \\ & + C_1 \left( \left\| \nabla^l u(t) \right\|_{H^{N-l}}^2 + \left\| \nabla^{l+1} \Theta(t) \right\|_{H^{N-l}}^2 \right) \\ & \leq C_2 \delta \left\| \nabla^{l+1} \mathbf{q}(t) \right\|_{H^{N-l-1}}^2. \end{aligned} \quad (70)$$

Summing up (60) of Lemma 19 from  $k=l$  to  $k=N-1$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \int \sum_{k=l}^{N-1} \nabla^k u \cdot \nabla \nabla^k \mathbf{q} + C_3 \left\| \nabla^{l+1} \mathbf{q}(t) \right\|_{H^{N-l-1}}^2 \\ & \leq C_4 \left( \left\| \nabla^l u(t) \right\|_{H^{N-l}}^2 + \left\| \nabla^{l+1} \Theta(t) \right\|_{H^{N-l-1}}^2 \right). \end{aligned} \quad (71)$$

Multiplying (71) by  $2C_2\delta/C_3$  and then adding it to (70), since  $\delta$  is small, we deduce

$$\begin{aligned} & \frac{d}{dt} \left[ \left\| \nabla^l(\mathbf{q}, u, \Theta)(t) \right\|_{H^{N-l-1}}^2 + \int \mathfrak{E}^N(t) + \frac{2C_2\delta}{C_3} \int \sum_{k=l}^{N-1} \nabla^k u \cdot \nabla \nabla^k \mathbf{q} \right] \\ & + C_5 \left( \left\| \nabla^{l+1} \mathbf{q}(t) \right\|_{H^{N-l-1}}^2 + \left\| \nabla^l u(t) \right\|_{H^{N-l}}^2 + \left\| \nabla^{l+1} \Theta(t) \right\|_{H^{N-l}}^2 \right) \\ & \leq 0. \end{aligned} \quad (72)$$

We define

$$\begin{aligned} \mathcal{E}_l^N(t) & := \frac{1}{C_5} \left[ \left\| \nabla^l(\mathbf{q}, u, \Theta)(t) \right\|_{H^{N-l-1}}^2 + \int \mathfrak{E}^N(t) \right. \\ & \left. + \frac{2C_2\delta}{C_3} \int \sum_{k=l}^{N-1} \nabla^k u \cdot \nabla \nabla^k \mathbf{q} \right]. \end{aligned} \quad (73)$$

Note that

$$\mathfrak{E}^N(t) = \frac{1+\Theta}{1+\Theta} |\nabla^N \mathbf{q}|^2 + (1+\Theta) |\nabla^{N-1} \operatorname{div} u|^2 + |\nabla^{N-1} \operatorname{curl} u|^2 + \frac{1+\Theta}{1+\Theta} |\nabla^N \Theta|^2. \quad (74)$$

By (28) and (56), since  $\delta$  is small, we can deduce from (73) to (74) that there exists a positive constant  $c$  such that for any  $t \in [0, T]$ ,

$$\frac{1}{c} \left\| \nabla^l(\mathbf{q}, u, \Theta)(t) \right\|_{H^{N-l}}^2 \leq \mathcal{E}_l^N(t) \leq c \left\| \nabla^l(\mathbf{q}, u, \Theta)(t) \right\|_{H^{N-l}}^2. \quad (75)$$

Hence, the proof of Lemma 20 is completed.

## 4. Global Solution

In this section, we will prove the existence and uniqueness of the global solution, namely, Theorem 1. We first record the local solution (cf. [52]).

**Proposition 21** (local-in-time solution). *Let  $l = 3$  or  $4$ . Assume that  $(\mathbf{q}_0, u_0, \Theta_0) \in H^l$  and  $\inf_{x \in \mathbb{R}^3} \{\mathbf{q}_0(x) + 1\} > 0$ . Then, there exists a constant  $T > 0$  such that the Cauchy problem (26) admits a unique solution  $(\mathbf{q}, u, \Theta)(t) \in \mathcal{E}(0, T; H^l)$  satisfying*

$$\begin{cases} \inf_{x \in \mathbb{R}^3, 0 \leq t \leq T} \{\mathbf{q}(x, t) + 1\} > 0, \\ \sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^l} \leq C_l \|(\mathbf{q}_0, u_0, \Theta_0)\|_{H^l}, \end{cases} \quad (76)$$

where  $C_l > 1$  is some fixed constant. Here,

$$\begin{aligned} \mathcal{E}(0, T; H^l) & := \{(\mathbf{q}, u, \Theta)(x, t) : (\mathbf{q}, u)(x, t) \\ & \in C^0(0, T; H^l) \cap C^1(0, T; H^{l-1}), \Theta(x, t) \\ & \in C^0(0, T; H^l) \cap C^1(0, T; H^{l-2})\}. \end{aligned} \quad (77)$$

Then, we construct the a priori estimates by using the energy estimates given in Lemma 20.

**Proposition 22** (a priori estimates). *Let  $N \geq 3$  and  $T > 0$ . Assume that for some sufficiently small  $\delta > 0$ ,*

$$\sup_{0 \leq t \leq T} \|(\mathbf{q}, u, \Theta)(t)\|_{H^3} < \delta, \quad \text{if } N = 3, \quad (78)$$

or

$$\sup_{0 \leq t \leq T} \|(\mathbf{Q}, u, \Theta)(t)\|_{H^4} < \delta, \quad \text{if } N \geq 4. \quad (79)$$

Then, we have for any  $t \in [0, T]$  and  $3 \leq \ell \leq N$ ,

$$\begin{aligned} & \|(\mathbf{Q}, u, \Theta)(t)\|_{H^\ell} + \left( \int_0^t (\|\nabla \mathbf{Q}(\tau)\|_{H^{\ell-1}}^2 + \|(u, \nabla \Theta)(\tau)\|_{H^\ell}^2) d\tau \right)^{1/2} \\ & \leq C_2 \|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^\ell}, \end{aligned} \quad (80)$$

where  $C_2 > 1$  is some fixed constant.

*Proof.* Let  $N \geq 3$  and  $3 \leq \ell \leq N$ . Letting  $l = 0$  and  $N = \ell$  in (67) of Lemma 20, we obtain

$$\frac{d}{dt} \mathcal{E}_0^\ell(t) + \|\nabla \mathbf{Q}(t)\|_{H^{\ell-1}}^2 + \|(u, \nabla \Theta)(t)\|_{H^\ell}^2 \leq 0. \quad (81)$$

Integrating (81) in time, we obtain

$$\mathcal{E}_0^\ell(t) + \int_0^t (\|\nabla \mathbf{Q}(\tau)\|_{H^{\ell-1}}^2 + \|(u, \nabla \Theta)(\tau)\|_{H^\ell}^2) d\tau \leq \mathcal{E}_0^\ell(0). \quad (82)$$

Letting  $l = 0$  and  $N = \ell$  in (75), we obtain for any  $t \in [0, T]$  and some  $c > 0$ ,

$$\frac{1}{c} \|(\mathbf{Q}, u, \Theta)(t)\|_{H^\ell}^2 \leq \mathcal{E}_0^\ell(t) \leq c \|(\mathbf{Q}, u, \Theta)(t)\|_{H^\ell}^2. \quad (83)$$

We immediately deduce (80) from (82) to (83).

Finally, we perform a continuous argument to extend the local solution given in Proposition 21 to the global one. We only consider the case  $N \geq 4$ . It is similar for  $N = 3$ .

Let  $N \geq 4$ . Assume  $(\mathbf{Q}_0, u_0, \Theta_0) \in H^N$  satisfying

$$\|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^4} < \frac{\delta}{C_1 C_2}, \quad (84)$$

where  $C_1 > 1, C_2 > 1, \delta > 0$  are given by Proposition 21 and Proposition 22. Since

$$\|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^4} < \frac{\delta}{C_1}, \quad (85)$$

by Proposition 21, there exists a constant  $T_1 > 0$  such that the Cauchy problem (26) has a unique local solution

$$(\mathbf{Q}, u, \Theta)(t) \in \mathcal{E}(0, T_1; H^4), \quad (86)$$

which satisfies

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{H^4} \leq C_1 \|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^4} < \delta, \quad \forall t \in [0, T_1]. \quad (87)$$

By (87) and Proposition 22, we obtain for any  $t \in [0, T_1]$  and  $3 \leq \ell \leq N$ ,

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{H^\ell} \leq C_2 \|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^\ell}, \quad (88)$$

which, together with (84), implies

$$(\mathbf{Q}, u, \Theta)(T_1) \in H^4, \quad \|(\mathbf{Q}, u, \Theta)(T_1)\|_{H^4} < \frac{\delta}{C_1}. \quad (89)$$

Then, choosing  $T_1 > 0$  as the new initial time instant, by Proposition 21 again, we obtain that the Cauchy problem (26) has a unique local solution

$$(\mathbf{Q}, u, \Theta)(t) \in \mathcal{E}(T_1, 2T_1; H^4), \quad (90)$$

such that

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{H^4} \leq C_1 \|(\mathbf{Q}, u, \Theta)(T_1)\|_{H^4} < \delta, \quad \forall t \in [T_1, 2T_1]. \quad (91)$$

From the above, we have proven that the Cauchy problem (26) has a unique local solution

$$(\mathbf{Q}, u, \Theta)(t) \in \mathcal{E}(0, 2T_1; H^4), \quad (92)$$

such that

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{H^4} < \delta, \quad \forall t \in [0, 2T_1]. \quad (93)$$

By (93) and Proposition 22, we obtain for any  $t \in [0, 2T_1]$  and  $3 \leq \ell \leq N$ ,

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{H^\ell} \leq C_2 \|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^\ell}, \quad (94)$$

which, together with (84) again, implies

$$(\mathbf{Q}, u, \Theta)(2T_1) \in H^4, \quad \|(\mathbf{Q}, u, \Theta)(2T_1)\|_{H^4} < \frac{\delta}{C_1}. \quad (95)$$

By repeating the above procedures, we can extend the local solution to the global one only if  $(\mathbf{Q}_0, u_0, \Theta_0) \in H^N$  ( $N \geq 4$ ) satisfying that  $\|(\mathbf{Q}_0, u_0, \Theta_0)\|_{H^4}$  is suitably small, as (84). So, we can choose  $\delta_0 = \delta/C_1 C_2$  in Theorem 1. Hence, the proof of Theorem 1 is completed.

## 5. Time-Decay Rates

In this section, we shall derive the time-decay rates (9) in Theorem 2. We first show that the negative Sobolev or Besov norms of the solution  $(\mathbf{Q}, u, \Theta)(t)$  can be controlled by the initial data.

**Lemma 23.** *Under the assumptions of Theorem 1, if further  $(\mathbf{Q}_0, u_0, \Theta_0) \in \dot{H}^{-s}$  for some  $s \in [0, 3/2)$ , or  $(\mathbf{Q}_0, u_0, \Theta_0) \in \dot{B}_{2,\infty}^{-s}$  for some  $s \in (0, 3/2]$ , then for all  $t \geq 0$ ,*

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{\dot{H}^{-s}} \leq C_0, \quad (96)$$

or

$$\|(\mathbf{Q}, u, \Theta)(t)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0. \quad (97)$$

*Proof.* As with [29], Theorem 1.2, (1.7)–(1.8), we omit the details.

To derive the time-decay rates of the solution and its higher-order derivatives, we prove the following differential inequality with respect to time.

**Lemma 24.** *Let  $N \geq 3$ . Under the assumptions of Theorem 1, it holds that for all  $t \geq 0$  and  $0 \leq l \leq N - 1$ ,*

$$\frac{d}{dt} \mathcal{E}_l^N(t) + \|\nabla^{l+1} \mathbf{Q}(t)\|_{H^{N-l-1}}^2 + \|\nabla^l u(t)\|_{H^{N-l}}^2 + \|\nabla^{l+1} \Theta(t)\|_{H^{N-l}}^2 \leq 0, \quad (98)$$

where

$$\mathcal{E}_l^N(t) \sim \|\nabla^l(\mathbf{Q}, u, \Theta)(t)\|_{H^{N-l}}^2. \quad (99)$$

*Proof.* It follows from Lemma 20 and Theorem 1.

Now, we can use Lemmas 23 and 24 to prove the time-decay rates (9) and (10) in Theorem 2.

For  $0 \leq l \leq N - 1$ , by Lemmas 15 and 16, we have

$$\|\nabla^{l+1} f\| \geq \|f\|_{\dot{H}^{-s}}^{-(1/(l+s))} \|\nabla^l f\|^{1+(1/(l+s))}, \quad (100)$$

$$\|\nabla^{l+1} f\| \geq \|f\|_{\dot{B}_{2,\infty}^{-s}}^{-(1/(l+s))} \|\nabla^l f\|^{1+(1/(l+s))}. \quad (101)$$

Combining Lemma 23 with (100) and (101), we have

$$\|\nabla^{l+1}(\mathbf{Q}, \Theta)\| \geq C_0 \|\nabla^l(\mathbf{Q}, \Theta)\|^{1+(1/(l+s))}. \quad (102)$$

This together with (8) infers for  $0 \leq l \leq N - 1$ ,

$$\begin{aligned} & \|\nabla^{l+1} \mathbf{Q}(t)\|_{H^{N-l-1}}^2 + \|\nabla^l u(t)\|_{H^{N-l}}^2 + \|\nabla^{l+1} \Theta(t)\|_{H^{N-l}}^2 \\ & \geq C_0 \left( \|\nabla^l(\mathbf{Q}, u, \Theta)\|_{H^{N-l}} \right)^{1+(1/(l+s))}. \end{aligned} \quad (103)$$

From the differential inequality (98) of Lemma 24, we obtain for  $0 \leq l \leq N - 1$ ,

$$\frac{d}{dt} \mathcal{E}_l^N(t) + C_0 (\mathcal{E}_l^N(t))^{1+(1/(l+s))} \leq 0. \quad (104)$$

Solving the above inequality directly, we get for  $0 \leq l \leq N - 1$ ,

$$\mathcal{E}_l^N(t) \leq C_0 (1+t)^{-(l+s)}. \quad (105)$$

By (99), we have for  $0 \leq l \leq N - 1$ ,

$$\|\nabla^l(\mathbf{Q}, u, \Theta)(t)\|_{H^{N-l}}^2 \leq C_0 (1+t)^{-(l+s)}. \quad (106)$$

*Higher decay of the velocity  $u$ .* Referring to (51), we know that the velocity  $u$  satisfies

$$u_t + u = -\nabla \mathbf{Q} - \nabla \Theta - u \cdot \nabla u + \frac{\mathbf{Q} - \Theta}{1 + \mathbf{Q}} \nabla \mathbf{Q}. \quad (107)$$

Let  $N \geq 3$  and  $0 \leq l \leq N - 2$ . Applying  $\nabla^l$  to (107), multiplying the resulting identity by  $\nabla^l u$  and integrating over  $\mathbb{R}^3$ , by Hölder's inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla^l u|^2 + \|\nabla^l u\|^2 \leq \|\nabla^l \left( \nabla \mathbf{Q}, \nabla \Theta, u \cdot \nabla u, \frac{\mathbf{Q} - \Theta}{1 + \mathbf{Q}} \nabla \mathbf{Q} \right)\| \|\nabla^l u\|, \quad (108)$$

which infers

$$\begin{aligned} \frac{d}{dt} \|\nabla^l u\|^2 + \|\nabla^l u\|^2 & \leq \|\nabla^l \left( \nabla \mathbf{Q}, \nabla \Theta, u \cdot \nabla u, \frac{\mathbf{Q} - \Theta}{1 + \mathbf{Q}} \nabla \mathbf{Q} \right)\|^2 \\ & \leq \|\nabla^{l+1}(\mathbf{Q}, \Theta)\|^2 + \|(\mathbf{Q}, u, \Theta)\|_{H^2}^2 \|\nabla^{l+1}(\mathbf{Q}, u, \Theta)\|^2 \\ & \leq C_0 (1+t)^{-(l+1+s)} \end{aligned} \quad (109)$$

by Cauchy's inequality, Lemmas 10 and 11, and (106). Applying Gronwall's inequality to (109), we obtain for  $0 \leq l \leq N - 2$ ,

$$\begin{aligned} \|\nabla^l u(t)\|^2 & \leq \|\nabla^l u_0\|^2 e^{-t} + C_0 \int_0^t e^{-(t-\tau)} (1+\tau)^{-(l+1+s)} d\tau \\ & \leq C_0 (1+t)^{-(l+1+s)}. \end{aligned} \quad (110)$$

Thus, we can deduce the decay rates (9) and (10) from (106) and (110). Hence, we complete the proof of Theorem 2.

### Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

### Conflicts of Interest

The authors declare that they have no conflict of interest.

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